Stochastic Processes and their Applications 99 (2002) 209–286

stochastic processes and their applications

www.elsevier.com/locate/spa

On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case

François Delarue

Projet SYSDYS (INRIA/LATP), Institut Méditerranéen de Technologie, 38, rue Joliot-Curie, F-13451 Marseille Cedex 20, France

Received 24 December 2000; received in revised form 16 November 2001; accepted 27 December 2001

Abstract

We prove a result of existence and uniqueness of solutions to forward–backward stochastic differential equations, with non-degeneracy of the diffusion matrix and boundedness of the coefficients as functions of x as main assumptions.

This result is proved in two steps. The first part studies the problem of existence and uniqueness over a small enough time duration, whereas the second one explains, by using the connection with quasi-linear parabolic system of PDEs, how we can deduce, from this local result, the existence and uniqueness of a solution over an arbitrarily prescribed time duration. Improving this method, we obtain a result of existence and uniqueness of classical solutions to non-degenerate quasi-linear parabolic systems of PDEs.

This approach relaxes the regularity assumptions required on the coefficients by the Four-Step scheme.

Keywords: Existence and uniqueness; Forward–backward stochastic differential equations; Gradient estimate; Quasi-linear equations of parabolic type

0. Introduction

$\forall N \in \mathbf{N}$, the notations $\langle , \rangle$ and $|\,|$ stand for the euclidian scalar product and the euclidian norm on $\mathbf{R}^N$.

Let T be a non-negative real, $(\Omega, \mathscr{A}, \mathbf{P})$ a probability space, and $(B_t)_{0 \leqslant t \leqslant T}$ a P-dimensional Brownian motion, whose natural filtration, augmented with $\mathscr{N}$ is denoted by $\{\mathscr{F}_t\}_{0 \leqslant t \leqslant T}$, where

$$\mathscr{N} = \{F \subset \Omega;\ \exists G \in \mathscr{A},\ F \subset G,\ \mathbf{P}(G) = 0\}.$$

E-mail address: fdelarue@sophia.inria.fr (F. Delarue).

0304-4149/02/$ - see front matter
PII: S0304-4149(02)00085-6

Let us consider in addition a filtration $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$, satisfying the usual conditions, and such that $(B_t)_{0\leqslant t\leqslant T}$ is still a $\{\mathscr{G}_t\}$-Brownian motion. (So, we have $\forall t\in[0,T]$, $\mathscr{F}_t\subset\mathscr{G}_t$.)

Moreover, let

$$
\text{(A1.0)}:\quad
\begin{aligned}
&f:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}\to\mathbf{R}^P,\\
&g:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}\to\mathbf{R}^Q,\\
&\sigma:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to\mathbf{R}^{P\times P},\\
&h:\mathbf{R}^P\to\mathbf{R}^Q,
\end{aligned}
$$

be measurable functions with respect to the borelian σ-fields.

For any $\mathbf{R}^P$ valued and $\mathscr{G}_0$-measurable random vector ξ, satisfying $\mathbf{E}|\xi|^2<\infty$, we are seeking an $\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$ valued and $\{\mathscr{G}_t\}$-progressively measurable process $(X_t,Y_t,Z_t)_{0\leqslant t\leqslant T}$, solution of the problem:

$$
\text{(E)}\quad
\begin{cases}
\forall t\in[0,T],\\
X_t=\xi+\displaystyle\int_0^t f(s,X_s,Y_s,Z_s)\,\mathrm{d}s+\int_0^t\sigma(s,X_s,Y_s)\,\mathrm{d}B_s,\\
Y_t=h(X_T)+\displaystyle\int_t^T g(s,X_s,Y_s,Z_s)\,\mathrm{d}s-\int_t^T Z_s\,\mathrm{d}B_s,\\
\mathbf{E}\displaystyle\int_0^T(|X_t|^2+|Y_t|^2+|Z_t|^2)\,\mathrm{d}t<\infty.
\end{cases}
$$

The aim of the following work is to present sufficient conditions on the coefficients of the FBSDE (E) to insure existence and uniqueness of a solution to such an equation. In fact, having chosen deterministic coefficients f, g, h and σ allows us to use the very strong link which exists between such kind of FBSDE and quasi-linear parabolic systems of PDEs: to the problem (E) is associated the system:

$$
\text{(E}'\text{)}\quad
\begin{cases}
\forall(t,x)\in[0,T]\times\mathbf{R}^P,\quad\forall\ell\in\{1,..,Q\},\\
\dfrac{\partial\theta_\ell}{\partial t}(t,x)+\dfrac{1}{2}\displaystyle\sum_{i,j=1}^P a_{i,j}(t,x,\theta(t,x))\frac{\partial^2\theta_\ell}{\partial x_i\,\partial x_j}(t,x)\\
\quad+\displaystyle\sum_{i=1}^P f_i(t,x,\theta(t,x),\nabla_x\theta(t,x)\sigma(t,x,\theta(t,x)))\frac{\partial\theta_\ell}{\partial x_i}(t,x)\\
\quad+g_\ell(t,x,\theta(t,x),\nabla_x\theta(t,x)\sigma(t,x,\theta(t,x)))=0,\\
\forall x\in\mathbf{R}^P,\quad\theta(T,x)=h(x),
\end{cases}
$$

with,

$$\forall(t,x,y)\in[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q,\quad a(t,x,y)=\sigma\sigma^*(t,x,y).$$

Several articles have already studied or used this connection between FBSDEs and quasi-linear PDEs.

For example, inspired by the earlier work of Ma and Yong (1995) and widely using the results of Ladyzenskaja et al. (1968) on systems of type (E$'$), Ma et al. (1994) have introduced the so called *four step scheme* and proved that, under quite strong regularity assumptions on the coefficients and non-degeneracy of the diffusion coefficient of the forward equation, the problem (E) admits a unique solution. Recently, in the same spirit, Hu and Yong (2000) have relaxed these regularity assumptions to obtain an existence (but not uniqueness) result for a subclass of the FBSDEs of type (E) (in particular, they assume f to be independent of z).

Considering systems of type (E) with random coefficients, Pardoux and Tang (1999) have established, by means of a purely probabilistic approach, and as soon as the coefficients satisfy a quite simple monotonicity condition, an existence and uniqueness result. This case allows the matrix σ to be degenerate. Moreover, the authors have deduced how, under appropriate assumptions, the solution of the problem (E) provides a viscosity solution to the problem (E$'$), and therefore have introduced a new way of connecting FBSDEs with quasi-linear PDEs.

Several other results give sufficient conditions to insure the unique solvability of FBSDEs of a more general form (in such cases, the matrix σ is allowed to depend on z). Among them, we mention Hu and Peng (1995) as well as Peng and Wu (1999) who have established an existence and uniqueness result under certain monotonicity conditions (different from the one of Pardoux and Tang, 1999). Observing these latter approaches, Yong (1997) has introduced the notion of *bridge* in order to compare in a very general way the nature (i.e. the solvability) of two different equations. Applying this technique, the author recovers the cases treated by Hu and Peng (1995) and Peng and Wu (1999), and obtains some results on linear systems. Using some functional analysis, these latter ones are extended in Yong (1999).

Finally, let us note that some of these results are also presented in the recent monograph on the subject by Ma and Yong (1999).

The approach, which is presented here, deals with the non-degenerate case. It is based both on probabilistic techniques and on some PDE results. We proceed in two steps.

We firstly give, (see also Antonelli, 1993; Pardoux and Tang, 1999), under quite classical assumptions (monotonous-Lipschitzian coefficients) and by means of a fixed point theorem, a result of existence and uniqueness in the case of a small enough time duration T (Theorem 1.1). We then aim to extend by means of an induction this local result to a global one. Actually, as explained in our paper, the crucial point of such a method is the control of the length of the interval on which Theorem 1.1 insures us existence and uniqueness of the solution.

Then, using some estimates of the gradient of solutions of quasi-linear parabolic systems of PDEs (see Ladyzenskaja et al., 1968), we prove that under appropriate assumptions (non-degeneracy of σ and boundedness of the coefficients as functions of x) this control is efficient: this means that, when extending with an induction the small time duration result to a large time duration one, this length does not become too small. We finally obtain a global existence and uniqueness theorem (Theorem 2.6), which certainly is our main result. By the way, let us note that Hu and Yong (2000) have used the same kind of PDE estimates to establish their existence result under more restrictive assumptions than ours, which constitutes in the end and roughly speaking one of the two sides of our theorem.

Actually, our approach also improves the result of Ma et al. (1994), by relaxing the regularity conditions that the *four step scheme* requires. In fact, we feel, as soon as the matrix σ is non-degenerate, that the assumptions needed in our theorem to insure the unique solvability of the problem (E) are rather weak, and in the end seem quite natural when compared with the usual result for classical SDEs. Moreover, we think that the iterative scheme that we expose in this paper to enlarge the local result to a global one, is quite flexible and could be applied to other classes of FBSDEs.

Our paper is organized as follows. Section 1 is devoted to the proof of existence and uniqueness of solutions over a small enough time duration. We also present some properties of the solutions, like continuous dependence upon the initial conditions and upon the coefficients. In Section 2, we prove our main result (existence and uniqueness under a non-degeneracy assumption) by showing how to deduce the global result from the theorem of local existence and uniqueness. Section 3 gives some extensions of the latter theorem: on one hand, we deal with a case where the coefficients are only locally monotonous-Lipschitzian, and on another one, we propose an extension of the uniqueness property to a larger set of non-standard FBSDEs.

Finally, we have added three appendices. In Appendices A and B, we deal with the regularity of the map $(t,x) \mapsto Y_t^{t,x}$. Using a method inspired by the one developed by Pardoux and Peng (1992) as well as PDE estimates of Ladyzenskaja et al. (1968), we deduce a theorem of existence and uniqueness of solutions to the system (E′) under a non-degeneracy assumption (this result is close to the one given in Ladyzenskaja et al., 1968). As another application, we prove, in Appendix C, a result of existence and uniqueness of solutions in a special case (essentially, $P = Q = 1$), which allows the diffusion coefficient to be degenerate.

1. Existence and uniqueness in small time duration

Assumption (A1). We say that the functions f, g, h and σ satisfy Assumption (A1) if there exist two constants K and Λ such that they satisfy both (A1.0) and the following properties:

(A1.1): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and $\forall (x',y',z') \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant K\ (|y-y'| + |z-z'|),$$

$$|g(t,x,y,z) - g(t,x',y,z')| \leqslant K\ (|x-x'| + |z-z'|),$$

$$|h(x) - h(x')| \leqslant K\ |x-x'|,$$

$$|\sigma(t,x,y) - \sigma(t,x',y')|^2 \leqslant K^2\ (|x-x'|^2 + |y-y'|^2).$$

(A1.2): $\forall t \in [0,T], \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and $\forall (x',y') \in \mathbf{R}^P \times \mathbf{R}^Q$,

$$\langle x-x', f(t,x,y,z) - f(t,x',y,z)\rangle \leqslant K\ |x-x'|^2,$$

$$\langle y-y', g(t,x,y,z) - g(t,x,y',z)\rangle \leqslant K\ |y-y'|^2.$$

(A1.3): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z)| \leqslant \Lambda\ (1+|x|+|y|+|z|),$$
$$|g(t,x,y,z)| \leqslant \Lambda\ (1+|x|+|y|+|z|),$$
$$|\sigma(t,x,y)| \leqslant \Lambda\ (1+|x|+|y|),$$
$$|h(x)| \leqslant \Lambda\ (1+|x|).$$

(A1.4): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, the functions $u \mapsto f(t,u,y,z)$ and $v \mapsto g(t,x,v,z)$ are continuous.

Theorem 1.1 (Existence and uniqueness in small time duration). *Assume that* (A1) *is in force. Then, for every $\mathscr{G}_0$-measurable random vector ξ, satisfying $\mathbf{E}|\xi|^2 < \infty$, every solution $(X_t, Y_t, Z_t)_{0\leqslant t\leqslant T}$ of the problem* (E) *satisfies*:

(i) $(X_t)_{0\leqslant t\leqslant T}$ *and* $(Y_t)_{0\leqslant t\leqslant T}$ *are continuous*;
(ii) $\mathbf{E}(\sup_{0\leqslant t\leqslant T}|X_t|^2 + \sup_{0\leqslant t\leqslant T}|Y_t|^2) < \infty$.

Moreover, there exists a constant $C_K^{(1)} > 0$, only depending on K, such that for every $T \leqslant C_K^{(1)}$, for every $\mathscr{G}_0$-measurable random vector ξ, satisfying $\mathbf{E}|\xi|^2 < \infty$, the problem (E) *admits a unique solution.*

Remark. This result cannot be extended to the more general case where:

$$\sigma : [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P} \to \mathbf{R}^{P\times P},$$
$$\forall t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}, \forall (x',y',z') \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P},$$
$$|\sigma(t,x,y,z) - \sigma(t,x',y',z')|^2 \leqslant K^2\ (|x-x'|^2 + |y-y'|^2 + |z-z'|^2),$$

i.e. to the case where σ explicitly depends on the variable z, and where the applications f, g, h, and σ only satisfy Lipschitz conditions. Indeed, we just have to consider the following problem:

$$\begin{cases} \forall t \in [0,T], \\ X_t = x + \displaystyle\int_0^t Z_s\,\mathrm{d}B_s, \\ Y_t = X_T - \displaystyle\int_t^T Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\displaystyle\int_0^T (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s < \infty \end{cases}$$

which has an infinite number of solutions.

Let us prove Theorem 1.1.

Proof. Let $(X_t, Y_t, Z_t)_{0\leqslant t\leqslant T}$ be a possible solution of the problem (E). Then, (i) is just a consequence of the property (A1.3) and of the inequality:

$$\mathbf{E}\int_0^T (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty.$$

Moreover, (ii) can be proved by using standard estimates and Burkholder–Davis–Gundy's inequalities.

Let us now prove existence and uniqueness of the solution in the case of a small enough T. Actually, we first show that there exists a unique $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$-progressively measurable solution, where:

$$\forall t \in [0,T], \quad \mathscr{G}_t^0 = \mathscr{G}_0 \vee \mathscr{F}_t.$$

We shall next prove that this solution is also the unique $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-progressively measurable solution.

Note that the filtration $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$ satisfies the usual conditions and that $(B_t)_{0\leqslant t\leqslant T}$ is a $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$-Brownian motion. Moreover, thanks to Theorem 4.33 of Chapter III (p. 176) of Jacod and Shiryaev (1987), we know that every $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$ local martingale can be represented as a stochastic integral with respect to B.

For the moment, let us consider for a natural integer N the following spaces:

$$\begin{cases} H_T^2(\mathbf{R}^N) & \text{space of } \{\mathscr{G}_t^0\}\text{-progressively measurable processes} \\ & \Phi : \Omega \times [0,t] \to \mathbf{R}^N \mid \|\Phi\|_2^2 = \mathbf{E}\int_0^T |\Phi_t|^2\,\mathrm{d}t < \infty, \\ S_T^2(\mathbf{R}^N) & \text{space of continuous } \{\mathscr{G}_t^0\}\text{-adapted processes} \\ & \Phi : \Omega \times [0,T] \to \mathbf{R}^N \mid \|\Phi\|_{2,*}^2 = \mathbf{E}\,\sup_{[0,T]} |\Phi_t|^2 < \infty. \end{cases}$$

We consider the map:

$$\Xi : S_T^2(\mathbf{R}^P) \times S_T^2(\mathbf{R}^Q) \times H_T^2(\mathbf{R}^{Q\times P}) \to S_T^2(\mathbf{R}^P) \times S_T^2(\mathbf{R}^Q) \times H_T^2(\mathbf{R}^{Q\times P})$$
$$(X,Y,Z) \mapsto (\bar{X},\bar{Y},\bar{Z}),$$

where $(\bar{X},\bar{Y},\bar{Z})$ is defined as follows:

$$\begin{cases} \forall t \in [0,T], \\ \bar{X}_t = \xi + \int_0^t f(s,\bar{X}_s,Y_s,Z_s)\,\mathrm{d}s + \int_0^t \sigma(s,\bar{X}_s,Y_s)\,\mathrm{d}B_s, \end{cases}$$

and

$$\begin{cases} \forall t \in [0,T], \\ \bar{Y}_t = h(\bar{X}_T) + \int_t^T g(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s)\,\mathrm{d}s - \int_t^T \bar{Z}_s\,\mathrm{d}B_s, \\ \mathbf{E}\int_0^T |\bar{Z}_s|^2\,\mathrm{d}s < \infty. \end{cases}$$

The process $(\bar{X}_t)_{0\leqslant t\leqslant T}$ is built as a solution of a forward SDE, whereas the couple $(\bar{Y}_t, \bar{Z}_t)_{t\in[0,T]}$ is built as a solution of a backward SDE.

In fact, solutions of BSDEs are usually built with respect to the couple $((B_t)_{0\leqslant t\leqslant T}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T})$ (see for example Pardoux, 1999). However, thanks to the choice of $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$, we have seen that the representation theorem is still valid with respect to the couple $((B_t)_{0\leqslant t\leqslant T}, \{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T})$. Hence, we let the reader see that the theorem of existence and uniqueness of solutions to BSDEs given in Pardoux (1999) can be extended to our case.

Actually, we want to prove that there exists a constant $C_K^{(1)} > 0$, only depending on K, such that for $T \leqslant C_K^{(1)}$, Ξ is a contraction. To this end, we firstly assume that $T \leqslant 1$, and we consider (X,Y,Z) and (U,V,W) in $S_T^2(\mathbf{R}^P)\times S_T^2(\mathbf{R}^Q)\times H_T^2(\mathbf{R}^{Q\times P})$. We put

$$(\bar{X},\bar{Y},\bar{Z}) = \Xi(X,Y,Z), \quad (\bar{U},\bar{V},\bar{W}) = \Xi(U,V,W).$$

Hence, from Itô's formula and Assumption (A1), there exists a constant γ_K, only depending on K, such that

$$\begin{aligned}\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{X}_t-\bar{U}_t|^2 \leqslant \gamma_K \Bigg[&\mathbf{E}\int_0^T (|\bar{X}_s-\bar{U}_s|(|\bar{X}_s-\bar{U}_s|+|Y_s-V_s|+|Z_s-W_s|))\,\mathrm{d}s\\ &+\mathbf{E}\int_0^T (|\bar{X}_s-\bar{U}_s|^2+|Y_s-V_s|^2)\,\mathrm{d}s\Bigg]\\ &+2\mathbf{E}\sup_{[0,T]}\left|\int_0^t \langle \bar{X}_s-\bar{U}_s, (\sigma(s,\bar{X}_s,Y_s)-\sigma(s,\bar{U}_s,V_s))\,\mathrm{d}B_s\rangle\right|.\end{aligned}$$

Using Burkholder–Davis–Gundy's inequalities, and modifying γ_K if necessary,

$$\begin{aligned}\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{X}_t-\bar{U}_t|^2 \leqslant \gamma_K \Bigg[&\mathbf{E}\int_0^T (|\bar{X}_s-\bar{U}_s|(|\bar{X}_s-\bar{U}_s|+|Y_s-V_s|+|Z_s-W_s|))\,\mathrm{d}s\\ &+\mathbf{E}\int_0^T (|\bar{X}_s-\bar{U}_s|^2+|Y_s-V_s|^2)\,\mathrm{d}s\\ &+\mathbf{E}\left(\int_0^T |\bar{X}_s-\bar{U}_s|^2(|\bar{X}_s-\bar{U}_s|^2+|Y_s-V_s|^2)\,\mathrm{d}s\right)^{1/2}\Bigg].\end{aligned}$$

Using standard estimates, and modifying γ_K once again if necessary, we have

$$\begin{aligned}&\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{X}_t-\bar{U}_t|^2\\ &\quad\leqslant \gamma_K T^{1/2}\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_s-\bar{U}_s|^2+\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s-V_s|^2+\mathbf{E}\int_0^T|Z_s-W_s|^2\,\mathrm{d}s\right).\end{aligned}$$

In particular,

$$\begin{aligned}&(1-\gamma_K T^{1/2})\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{X}_t-\bar{U}_t|^2\\ &\quad\leqslant \gamma_K T^{1/2}\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s-V_s|^2+\mathbf{E}\int_0^T|Z_s-W_s|^2\,\mathrm{d}s\right). \qquad (1.1.1)\end{aligned}$$

Moreover, Itô's formula shows that $\forall t \in [0,T]$,

$$|\bar{Y}_t - \bar{V}_t|^2 + \int_t^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s$$
$$= |h(\bar{X}_T) - h(\bar{U}_T)|^2 + 2\int_t^T \langle \bar{Y}_s - \bar{V}_s, g(s,\bar{X}_s,\bar{Y}_s,\bar{Z}_s) - g(s,\bar{U}_s,\bar{V}_s,\bar{W}_s)\rangle \,\mathrm{d}s$$
$$- 2\int_t^T \langle \bar{Y}_s - \bar{V}_s, (\bar{Z}_s - \bar{W}_s)\,\mathrm{d}B_s\rangle. \tag{1.1.2}$$

Now, the estimate

$$\mathbf{E}\left(\int_0^T |\bar{Y}_s - \bar{V}_s|^2 |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s\right)^{1/2}$$
$$\leqslant \mathbf{E}\left(\sup_{0\leqslant s\leqslant T} |\bar{Y}_s - \bar{V}_s|^2 + \int_0^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s\right) < \infty,$$

proves from Burkholder–Davis–Gundy's inequalities that

$$\forall t \in [0,T], \quad \mathbf{E}\int_t^T \langle \bar{Y}_s - \bar{V}_s, (\bar{Z}_s - \bar{W}_s)\,\mathrm{d}B_s\rangle = 0$$

and implies, using Assumption (A1), that there exists a constant γ'_K, only depending on K, such that:

$$\mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s \leqslant \gamma'_K \Bigg[\mathbf{E}|\bar{X}_T - \bar{U}_T|^2$$
$$+ \mathbf{E}\int_0^T (|\bar{Y}_s - \bar{V}_s|(|\bar{X}_s - \bar{U}_s| + |\bar{Y}_s - \bar{V}_s| + |\bar{Z}_s - \bar{W}_s|))\,\mathrm{d}s\Bigg].$$

In particular, modifying γ'_K if necessary,

$$\mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s \leqslant \gamma'_K \left((1+T)\mathbf{E}\sup_{0\leqslant s\leqslant T} |\bar{X}_s - \bar{U}_s|^2 + T\,\mathbf{E}\sup_{0\leqslant s\leqslant T} |\bar{Y}_s - \bar{V}_s|^2\right)$$
$$+ \frac{1}{2}\mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s.$$

By modifying γ'_K once again if necessary, we deduce

$$\mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2 \,\mathrm{d}s$$
$$\leqslant \gamma'_K \left((1+T)\mathbf{E}\sup_{0\leqslant s\leqslant T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E}\sup_{0\leqslant s\leqslant T} |\bar{Y}_s - \bar{V}_s|^2\right). \tag{1.1.3}$$

Moreover, considering (1.1.2), and still using Burkholder–Davis–Gundy's inequalities, we show that there exists a constant γ''_K, only depending on K, such that

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{Y}_t-\bar{V}_t|^2 \leqslant \gamma''_K\left[\mathbf{E}|\bar{X}_T-\bar{U}_T|^2+\mathbf{E}\left(\int_0^T|\bar{Y}_s-\bar{V}_s|^2|\bar{Z}_s-\bar{W}_s|^2\,\mathrm{d}s\right)^{1/2}\right.$$

$$\left.+\mathbf{E}\int_0^T(|\bar{Y}_s-\bar{V}_s|(|\bar{X}_s-\bar{U}_s|+|\bar{Y}_s-\bar{V}_s|+|\bar{Z}_s-\bar{W}_s|))\,\mathrm{d}s\right].$$

Hence, using (1.1.3), and modifying γ''_K if necessary,

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{Y}_t-\bar{V}_t|^2 \leqslant \gamma''_K\left((1+T)\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_s-\bar{U}_s|^2+T\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{Y}_s-\bar{V}_s|^2\right)$$

$$+\frac{1}{2}\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{Y}_s-\bar{V}_s|^2.$$

By modifying γ''_K if necessary, this proves that:

$$(1-\gamma''_K T)\,\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{Y}_t-\bar{V}_t|^2 \leqslant \gamma''_K(1+T)\,\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_s-\bar{U}_s|^2. \tag{1.1.4}$$

So, considering (1.1.1), (1.1.3) and (1.1.4), we obtain the three following inequalities:

$$(1-\gamma_K T^{1/2})\,\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_t-\bar{U}_t|^2$$

$$\leqslant \gamma_K T^{1/2}\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s-V_s|^2+\mathbf{E}\int_0^T|Z_s-W_s|^2\,\mathrm{d}s\right),$$

$$(1-\gamma''_K T)\,\mathbf{E}\sup_{0\leqslant t\leqslant T}|\bar{Y}_t-\bar{V}_t|^2 \leqslant \gamma''_K(1+T)\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_s-\bar{U}_s|^2,$$

$$\mathbf{E}\int_0^T|\bar{Z}_s-\bar{W}_s|^2\,\mathrm{d}s$$

$$\leqslant \gamma'_K\left((1+T)\,\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{X}_s-\bar{U}_s|^2+T\mathbf{E}\sup_{0\leqslant s\leqslant T}|\bar{Y}_s-\bar{V}_s|^2\right).$$

This proves that there exists a constant $C_K^{(1)}>0$ only depending on K, such that for $T\leqslant C_K^{(1)}$, the map Ξ is contractive from $S_T^2(\mathbf{R}^P)\times S_T^2(\mathbf{R}^Q)\times H_T^2(\mathbf{R}^{Q\times P})$ into itself. Consequently, the Picard's fixed point theorem shows that, for every $T\leqslant C_K^{(1)}$, there is a unique $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$-progressively measurable solution to the problem (E), denoted $(X_t^0,Y_t^0,Z_t^0)_{0\leqslant t\leqslant T}$. This solution is obviously $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-progressively measurable.

Actually, considering once again the estimates used to establish that the map Ξ is contractive (*the particular choice of the filtration* $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$ *is just needed for the construction of* Ξ), we prove that for every $T\leqslant C_K^{(1)}$, $(X_t^0,Y_t^0,Z_t^0)_{0\leqslant t\leqslant T}$ is also the unique $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-progressively measurable solution of the problem (E). □

Remark 1.2. Theorem 1.1 shows that for every $T \leqslant C_K^{(1)}$, and for every $x \in \mathbf{R}^P$, the problem:

$$\begin{cases} \forall t \in [0,T], \\ X_t = x + \int_0^t f(s,X_s,Y_s,Z_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s,Y_s)\,\mathrm{d}B_s, \\ Y_t = h(X_T) + \int_t^T g(s,X_s,Y_s,Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\int_0^T (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}s < \infty \end{cases}$$

admits a unique $\{\mathscr{F}_t\}$-progressively measurable solution $(X_t, Y_t, Z_t)_{t\in[0,T]}$. In particular, Y_0 is an $\mathscr{F}_0$-measurable random vector, and therefore is deterministic.

Theorem 1.3 (Main estimate). *Suppose that* (A1) *is in force. Then, there exists a constant* $0 < C_K^{(2)} \leqslant C_K^{(1)}$ *only depending on K, such that for every* $T \leqslant C_K^{(2)}$, *for every vector of functions* $(\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma})$ *satisfying Assumption* (A1) *with the same constants K and* Λ *as* (f,g,h,σ), *for every* $A \in \mathscr{G}_0$, *and for all* $\mathscr{G}_0$*-measurable random vectors* ξ *and* $\tilde{\xi}$, *with finite second moment, we have the following estimate*:

$$\begin{aligned} &\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|X_s - \tilde{X}_s|^2\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|Y_s - \tilde{Y}_s|^2\right) + \mathbf{E}\int_0^T (\mathbf{1}_A|Z_s - \tilde{Z}_s|^2)\,\mathrm{d}s \\ &\quad\leqslant c_K^{(2)}\left[\mathbf{E}(\mathbf{1}_A|\xi-\tilde{\xi}|^2) + \mathbf{E}(\mathbf{1}_A|(h-\tilde{h})(X_T)|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\sigma-\tilde{\sigma}|^2(s,X_s,Y_s))\,\mathrm{d}s \right.\\ &\quad\quad \left. + \mathbf{E}\left(\int_0^T \mathbf{1}_A(|f-\tilde{f}| + |g-\tilde{g}|)(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^2\right], \end{aligned} \tag{1.3.1}$$

where $c_K^{(2)}$ *only depends on* K, *and where the processes* $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$ *and* $(\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)_{0\leqslant s\leqslant T}$ *stand for the solutions of the problems associated to the coefficients* (f,g,h,σ) *and* $(\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma})$ *and to the initial conditions* $(0,\xi)$ *and* $(0,\tilde{\xi})$.

Proof. We keep the notations given in the statement. Then, using Itô's calculus, we have

$$\begin{aligned} &\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T}|\tilde{X}_t - X_t|^2\right) \\ &\quad\leqslant \mathbf{E}(\mathbf{1}_A|\tilde{\xi}-\xi|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s)|^2)\,\mathrm{d}s \\ &\quad\quad + 2\mathbf{E}\sup_{0\leqslant t\leqslant T}\left(\int_0^t \mathbf{1}_A\langle \tilde{X}_s - X_s, \tilde{f}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - f(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\right) \\ &\quad\quad + 2\mathbf{E}\sup_{0\leqslant t\leqslant T}\left(\int_0^t \mathbf{1}_A\langle \tilde{X}_s - X_s, (\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s))\,\mathrm{d}B_s\rangle\right) \end{aligned}$$

so, using Burkholder–Davis–Gundy inequalities, there exists a constant γ such that

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{X}_t - X_t|^2\right)\\
&\quad\leqslant \mathbf{E}\,(\mathbf{1}_A|\tilde{\xi} - \xi|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s)|^2)\,\mathrm{d}s\\
&\qquad + 2\mathbf{E}\sup_{0\leqslant t\leqslant T}\left(\int_0^t \mathbf{1}_A\langle \tilde{X}_s - X_s, \tilde{f}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - f(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\right)\\
&\qquad + 2\gamma\mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{X}_s - X_s|^2|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s)|^2\,\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

In particular, using standard estimates, and modifying γ if necessary,

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{X}_t - X_t|^2\right)\\
&\quad\leqslant \gamma\left[\mathbf{E}(\mathbf{1}_A|\tilde{\xi} - \xi|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s)|^2)\,\mathrm{d}s\right.\\
&\qquad + \mathbf{E}\sup_{0\leqslant t\leqslant T}\left(\int_0^t \mathbf{1}_A\langle \tilde{X}_s - X_s, \tilde{f}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - \tilde{f}(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\right)\\
&\qquad \left. + \mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{f} - f|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^2\right].
\end{aligned}$$

Hence, using properties (A1.1) and (A1.2), there exists γ_K, only depending on K, such that

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{X}_t - X_t|^2\right) \leqslant \gamma_K\left[\mathbf{E}(\mathbf{1}_A|\tilde{\xi} - \xi|^2)\right.\\
&\quad + \mathbf{E}\int_0^T (\mathbf{1}_A(|\tilde{X}_s - X_s|^2 + |\tilde{Y}_s - Y_s|^2))\,\mathrm{d}s + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{X}_s - X_s||\tilde{Z}_s - Z_s|)\,\mathrm{d}s\\
&\quad \left. + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma} - \sigma|^2(s,X_s,Y_s))\,\mathrm{d}s + \mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{f} - f|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^2\right].
\end{aligned} \tag{1.3.2}$$

Moreover, for every $0 \leqslant t \leqslant T$,

$$\begin{aligned}
&\mathbf{E}(\mathbf{1}_A|\tilde{Y}_t - Y_t|^2) + \mathbf{E}\int_t^T (\mathbf{1}_A|\tilde{Z}_s - Z_s|^2)\,\mathrm{d}s\\
&\quad = \mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^2) + 2\mathbf{E}\int_t^T \mathbf{1}_A\langle \tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)\\
&\qquad - g(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s.
\end{aligned} \tag{1.3.3}$$

Furthermore, using Burkholder–Davis–Gundy inequalities, there exists a constant γ' such that

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{Y}_t - Y_t|^2\right) \\
&\quad\leqslant \mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^2) + \gamma'\mathbf{E}\left(\int_0^T \mathbf{1}_A|Y_s - \tilde{Y}_s|^2|Z_s - \tilde{Z}_s|^2\,\mathrm{d}s\right)^{1/2} \\
&\qquad + 2\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\int_t^T \mathbf{1}_A\langle \tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - g(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\right).
\end{aligned}$$

Hence, modifying γ' if necessary and using standard estimates as well as (1.3.3), we obtain

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{Y}_t - Y_t|^2\right) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{Z}_s - Z_s|^2)\,\mathrm{d}s \leqslant \gamma'\Bigg[\mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^2) \\
&\quad + \mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\int_t^T \mathbf{1}_A\langle \tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - \tilde{g}(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\right) \\
&\quad + \mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{g} - g|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^2\Bigg].
\end{aligned}$$

In particular, using (1.3.2) as well as properties (A1.1) and (A1.2), and modifying γ_K if necessary,

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{X}_t - X_t|^2\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{Y}_t - Y_t|^2\right) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{Z}_s - Z_s|^2)\,\mathrm{d}s \\
&\quad\leqslant \gamma_K\Bigg[\mathbf{E}(\mathbf{1}_A|\tilde{\xi} - \xi|^2) + \mathbf{E}(\mathbf{1}_A|(\tilde{h} - h)(X_T)|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{X}_s - X_s|^2)\,\mathrm{d}s \\
&\qquad + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{Y}_s - Y_s|^2)\,\mathrm{d}s \\
&\qquad + \mathbf{E}\left(\int_0^T \mathbf{1}_A(|\tilde{f} - f| + |\tilde{g} - g|)(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^2 \\
&\qquad + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma} - \sigma|^2(s,X_s,Y_s))\,\mathrm{d}s \\
&\qquad + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{Z}_s - Z_s|(|\tilde{Y}_s - Y_s| + |\tilde{X}_s - X_s|))\,\mathrm{d}s\Bigg]. \qquad (1.3.4)
\end{aligned}$$

Hence, there exist two constants $c'_K > 0$ and γ'_K, only depending on K, such that for every $T \leqslant c'_K$,

$$\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{X}_t - X_t|^2\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant t\leqslant T} |\tilde{Y}_t - Y_t|^2\right) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{Z}_s - Z_s|^2)\,\mathrm{d}s$$

$$\leqslant \gamma'_K \left[\mathbf{E}(\mathbf{1}_A|\tilde{\xi} - \xi|^2) + \mathbf{E}(\mathbf{1}_A|(\tilde{h} - h)(X_T)|^2) + \mathbf{E}\int_0^T (\mathbf{1}_A|\tilde{\sigma} - \sigma|^2(s, X_s, Y_s))\,\mathrm{d}s\right.$$

$$\left. + \mathbf{E}\left(\int_0^T \mathbf{1}_A(|\tilde{f} - f| + |\tilde{g} - g|)(s, X_s, Y_s, Z_s)\,\mathrm{d}s\right)^2\right]. \tag{1.3.5}$$

□

Corollary 1.4 (Dependence upon the initial conditions). *Suppose that* (A1) *is in force. For every $T \leqslant C_K^{(2)}$, for every $t \in [0,T]$, and for every $\mathcal{G}_t$-measurable random vector ξ, with finite second moment, we define the process $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{t\leqslant s\leqslant T}$ as the unique solution of the problem*

$$\begin{cases} \forall s \in [t,T], \\ X_s = \xi + \displaystyle\int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^T (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s < \infty, \end{cases}$$

extended to the whole interval $[0,T]$ if $\xi = x$ a.s., $x \in \mathbf{R}^P$, by putting

$$\forall\ 0 \leqslant s \leqslant t, \quad X_s^{t,x} = x, \quad Y_s^{t,x} = Y_t^{t,x}, \quad Z_s^{t,x} = 0.$$

Then, the following properties are satisfied:
(1.4.1) *There exists a constant $c_{K,\Lambda}^{(3.1)}$ only depending on K and Λ such that $\forall (t,x) \in [0,T] \times \mathbf{R}^P$,*

$$\mathbf{E}\sup_{0\leqslant s\leqslant T} |X_s^{t,x}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T} |Y_s^{t,x}|^2 + \mathbf{E}\int_0^T |Z_s^{t,x}|^2\,\mathrm{d}s \leqslant c_{K,\Lambda}^{(3.1)}(1 + |x|^2).$$

(1.4.2) *There exists a constant $c_{K,\Lambda}^{(3.2)}$ only depending on K and Λ such that $\forall (t,t') \in [0,T]^2$, $\forall (x,x') \in (\mathbf{R}^P)^2$,*

$$\mathbf{E}\sup_{0\leqslant s\leqslant T} |X_s^{t',x'} - X_s^{t,x}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T} |Y_s^{t',x'} - Y_s^{t,x}|^2 + \mathbf{E}\int_0^T |Z_s^{t',x'} - Z_s^{t,x}|^2\,\mathrm{d}s$$

$$\leqslant c_K^{(2)}|x - x'|^2 + c_{K,\Lambda}^{(3.2)}(1 + |x|^2)|t' - t|.$$

Proof. Let us assume that $T \leqslant C_K^{(2)}$. Let us prove (1.4.1). Noting that for every $(t,x) \in [0,T] \times \mathbf{R}^P$ the process $(Y_s^{t,x})_{0\leqslant s\leqslant t}$ is actually reduced to a deterministic vector, we see that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0\leqslant s\leqslant T}$ is the solution of the problem:

$$\begin{cases} \forall s\in[0,T], \\ X_s = x + \displaystyle\int_0^s \mathbf{1}_{[t,T]}(r) f(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_0^s \mathbf{1}_{[t,T]}(r)\sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T \mathbf{1}_{[t,T]}(r) g(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_0^T (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty. \end{cases} \tag{1.4.3}$$

Then, by noting that the vectors of functions $(\mathbf{1}_{[t,T]}f, \mathbf{1}_{[t,T]}g, \mathbf{1}_{[t,T]}\sigma, h)$ and $(0,0,0,0)$ satisfy Assumption (A1), Theorem 1.3 shows that

$$\begin{aligned} &\mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s^{t,x}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}|^2 + \mathbf{E}\int_0^T |Z_s^{t,x}|^2\,\mathrm{d}s \\ &\quad\leqslant c_K^{(2)}\left[|x|^2 + |h(0)|^2 + \left(\int_0^T (|f|+|g|)(s,0,0,0)\,\mathrm{d}s\right)^2 + \int_0^T |\sigma|^2(s,0,0)\,\mathrm{d}s\right]. \end{aligned}$$

Then, property (A1.3) completes the proof of (1.4.1).

Let us prove (1.4.2). Let (t,t') be in $[0,T]^2$ and (x,x') in $(\mathbf{R}^P)^2$. Then, by noting once again that the vectors of functions $(\mathbf{1}_{[t,T]}f,\ \mathbf{1}_{[t,T]}g,\ \mathbf{1}_{[t,T]}\sigma, h)$ and $(\mathbf{1}_{[t',T]}f, \mathbf{1}_{[t',T]}g, \mathbf{1}_{[t',T]}\sigma, h)$ satisfy Assumption (A1), Theorem 1.3 and Eq. (1.4.3) show that

$$\begin{aligned} &\mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s^{t',x'} - X_s^{t,x}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s^{t',x'} - Y_s^{t,x}|^2 + \mathbf{E}\int_0^T |Z_s^{t',x'} - Z_s^{t,x}|^2\,\mathrm{d}s \\ &\quad\leqslant c_K^{(2)}\left[|x-x'|^2 + \mathbf{E}\left(\int_{t\wedge t'}^{t\vee t'} (|f|+|g|)(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,\mathrm{d}s\right)^2\right. \\ &\qquad\left. + \mathbf{E}\left(\int_{t\wedge t'}^{t\vee t'} |\sigma|^2(s,X_s^{t,x},Y_s^{t,x})\,\mathrm{d}s\right)\right]. \end{aligned}$$

Therefore, the property (A1.3) as well as (1.4.1) prove that there exists a constant $c_{K,\Lambda}^{(3.2)}$ only depending on K and Λ such that

$$\begin{aligned} &\mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s^{t',x'} - X_s^{t,x}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s^{t',x'} - Y_s^{t,x}|^2 + \mathbf{E}\int_0^T |Z_s^{t',x'} - Z_s^{t,x}|^2\,\mathrm{d}s \\ &\quad\leqslant c_K^{(2)}|x-x'|^2 + c_{K,\Lambda}^{(3.2)}(1+|x|^2)|t'-t|. \end{aligned}$$

This proves (1.4.2). □

Corollary 1.5. *Suppose that* (A1) *is in force and keep the notations of Corollary* 1.4. *Then, for every $T \leqslant C_k^{(2)}$, the map*

$$\theta : [0,T]\times\mathbf{R}^P \to \mathbf{R}^Q, \quad (t,x)\mapsto Y_t^{t,x},$$

satisfies $\forall (t,t') \in [0,T]^2$, $\forall (x,x') \in (\mathbf{R}^P)^2$,

$$|\theta(t,x)|^2 \leqslant c_{K,\Lambda}^{(3.1)}(1+|x|^2), \tag{1.5.1}$$

$$|\theta(t',x') - \theta(t,x)|^2 \leqslant c_K^{(2)}|x'-x|^2 + c_{K,\Lambda}^{(3.2)}(1+|x|^2)|t'-t|, \tag{1.5.2}$$

and for every $t \in [0,T]$, *for every* $\mathcal{G}_t$*-measurable random vector* ξ, *with finite second moment*, *there exists a* $\mathbf{P}$*-null set* $N_Y^{t,\xi} \in \mathcal{G}_0$, *such that*

$$\forall s \in [t,T], \quad \forall \omega \notin N_Y^{t,\xi}, \quad Y_s^{t,\xi}(\omega) = \theta(s, X_s^{t,\xi}(\omega)). \tag{1.5.3}$$

Proof. Let us consider $(t,x) \in [0,T] \times \mathbf{R}^P$. Then, from Remark 1.2, the vector $Y_t^{t,x}$ is deterministic, so that the map θ is correctly defined.

Moreover, (1.5.1) and (1.5.2) easily follow from (1.4.1) and (1.4.2).

Let us prove (1.5.3). Let ξ be a $\mathcal{G}_t$-measurable random vector with finite second moment. Theorem 1.3 shows that for every $\varepsilon > 0$,

$$\mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|Y_s^{t,\xi} - Y_s^{t,x}|^2) \leqslant c_K^{(2)}\mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|\xi-x|^2).$$

So, using Lipschitz property (1.5.2), we obtain

$$\begin{aligned}
&\mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|\theta(t,\xi) - Y_t^{t,\xi}|^2) \\
&\quad \leqslant 2[c_K^{(2)}\mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|\xi-x|^2) + \mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|\theta(t,\xi)-\theta(t,x)|^2)] \\
&\quad \leqslant 4c_K^{(2)}\mathbf{E}(\mathbf{1}_{\{|\xi-x|<\varepsilon\}}|\xi-x|^2).
\end{aligned}$$

Therefore, for any integer N,

$$\sum_{k\in\mathbf{Z}^P} \mathbf{E}(\mathbf{1}_{\{|\xi-k/N|_\infty<1/N\}}|\theta(t,\xi) - Y_t^{t,\xi}|^2) \leqslant \frac{4}{N^2}\, c_K^{(2)} \sum_{k\in\mathbf{Z}^P} \mathbf{E}(\mathbf{1}_{\{|\xi-k/N|_\infty<1/N\}}),$$

where $|\,|_\infty$ stands for the sup norm on $\mathbf{R}^P$. We deduce that for any integer N,

$$\mathbf{E}|\theta(t,\xi) - Y_t^{t,\xi}|^2 \leqslant \frac{2^{P+2}}{N^2}\, c_K^{(2)}.$$

In particular,

$$Y_t^{t,\xi} = \theta(t,\xi) \quad \text{a.s.} \tag{1.5.4}$$

Moreover, $\forall s \in [t,T]$, $(X_u^{t,\xi}, Y_u^{t,\xi}, Z_u^{t,\xi})_{s\leqslant u\leqslant T}$ is the solution of the problem

$$\begin{cases}
\forall u \in [s,T], \\
X_u = X_s^{t,\xi} + \displaystyle\int_s^u f(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_s^u \sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\
Y_u = h(X_T) + \displaystyle\int_u^T g(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_u^T Z_r\,\mathrm{d}B_r, \\
\mathbf{E}\displaystyle\int_s^T (|X_u|^2 + |Y_u|^2 + |Z_u|^2)\,\mathrm{d}u < \infty.
\end{cases}$$

So, (1.5.4) shows that

$$Y_u^{t,\xi} = \theta(u, X_u^{t,\xi}) \quad \text{a.s.}$$

The continuities of θ and of the trajectories of the processes $(X_s^{t,\xi})_{t \leqslant s \leqslant T}$ and $(Y_s^{t,\xi})_{t \leqslant s \leqslant T}$ show that, almost surely,

$$\forall u \in [t, T], \quad Y_u^{t,\xi} = \theta(u, X_u^{t,\xi}). \qquad \square$$

Remark 1.6. The map θ depends only on f, g, h, σ and T.

Proof. We use the same scheme as the one developed by Yamada and Watanabe to prove that pathwise uniqueness of solutions of SDE's implies uniqueness in the sense of probability law (see for example Karatzas and Shreve (1988), or Rogers and Williams (1987)).

Let us consider a real $0 < T \leqslant C_K^{(2)}$, two filtered probability spaces $(\Omega_i, \mathscr{A}_i, \{\mathscr{G}_t^{(i)}\}_{0 \leqslant t \leqslant T}, v_i)_{i=1,2}$, where $\{\mathscr{G}_t^{(1)}\}_{0 \leqslant t \leqslant T}$ and $\{\mathscr{G}_t^{(2)}\}_{0 \leqslant t \leqslant T}$ satisfy the usual conditions, and a $\{\mathscr{G}_t^{(1)}\}_{0 \leqslant t \leqslant T}$ (resp. $\{\mathscr{G}_t^{(2)}\}_{0 \leqslant t \leqslant T}$) Brownian motion, denoted $(W_t^{(1)})_{0 \leqslant t \leqslant T}$ (resp. $(W_t^{(2)})_{0 \leqslant t \leqslant T}$). We denote $\mathscr{E}_1$ (resp. $\mathscr{E}_2$) the expectation with respect to the probability measure v_1 (resp. v_2).

Let us also consider for $i \in \{1, 2\}$, $\xi^{(i)}$ a $\mathscr{G}_0^{(i)}$-measurable random vector, satisfying $\mathbf{E}\, |\xi^{(i)}|^2 < \infty$, and:

$$v(B) = v_1\{\xi^{(1)} \in B\} = v_2\{\xi^{(2)} \in B\}; \quad B \in \mathscr{B}(\mathbf{R}^P).$$

Finally, let us assume that for $i \in \{1, 2\}$, $(X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)})_{0 \leqslant t \leqslant T}$ is the solution of the problem

$$\begin{cases} \forall t \in [0, T], \\ X_t^{(i)} = \xi^{(i)} + \displaystyle\int_0^t f(s, X_s^{(i)}, Y_s^{(i)}, Z_s^{(i)})\, \mathrm{d}s + \int_0^t \sigma(s, X_s^{(i)}, Y_s^{(i)})\, \mathrm{d}W_s^{(i)}, \\ Y_t^{(i)} = h(X_T^{(i)}) + \displaystyle\int_t^T g(s, X_s^{(i)}, Y_s^{(i)}, Z_s^{(i)})\, \mathrm{d}s - \int_t^T Z_s^{(i)}\, \mathrm{d}W_s^{(i)}, \\ \mathbf{E} \displaystyle\int_0^T (|X_t^{(i)}|^2 + |Y_t^{(i)}|^2 + |Z_t^{(i)}|^2)\, \mathrm{d}t < \infty. \end{cases}$$

Let us also consider

$$\begin{aligned} (\Theta_T, \mathscr{B}(\Theta_T)) = (&\mathbf{C}([0, T], \mathbf{R}^P) \times \mathbf{C}([0, T], \mathbf{R}^Q) \times L^2([0, T], \mathbf{R}^{Q \times P}), \\ &\mathscr{B}(\mathbf{C}([0, T], \mathbf{R}^P)) \otimes \mathscr{B}(\mathbf{C}([0, T], \mathbf{R}^Q)) \otimes \mathscr{B}(L^2([0, T], \mathbf{R}^{Q \times P}))), \end{aligned}$$

as well as

$$(\Theta, \mathscr{B}(\Theta)) = (\mathbf{R}^P \times \mathbf{C}([0, T], \mathbf{R}^P) \times \Theta_T, \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{C}([0, T], \mathbf{R}^P)) \otimes \mathscr{B}(\Theta_T))$$

on which we can define two probability measures

$$\mathbf{P}_i(A) = v_i\{(\xi^{(i)}, W^{(i)}, X^{(i)}, Y^{(i)}, Z^{(i)}) \in A\}; \quad A \in \mathscr{B}(\Theta), \ i = 1, 2.$$

In particular, for $i \in \{1,2\}$,

$$\mathbf{P}_i(G_1 \times G_2 \times \Theta_T) = v(G_1)\mathbf{P}_*(G_2); \quad G_1 \in \mathscr{B}(\mathbf{R}^P), \;\; G_2 \in \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^P)),$$

where $\mathbf{P}_*$ stands for the Wiener measure on $(\mathbf{C}([0,T],\mathbf{R}^P), \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^P)))$.

Let us now define for $i \in \{1,2\}$,

$$Q_i(x,w;F) : \mathbf{R}^P \times \mathbf{C}([0,T],\mathbf{R}^P) \times \mathscr{B}(\Theta_T) \to [0,1]$$

as the regular conditional probability for $\mathscr{B}(\Theta_T)$ given (x,w) (under $\mathbf{P}_i$). It satisfies:

(i) $\forall (x,w) \in \mathbf{R}^P \times \mathbf{C}([0,T],\mathbf{R}^P)$, $Q_i(x,w,.)$ is a probability measure on $(\Theta_T, \mathscr{B}(\Theta_T))$.

(ii) $\forall F \in \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^P))$, the mapping $(x,w) \mapsto Q_i(x,w;F)$ is $\mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^P))$-measurable.

(iii) $\forall F \in \mathscr{B}(\Theta_T)$, $\forall G \in \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^P))$:

$$\mathbf{P}_i(G \times F) = \int_G Q_i(x,w,F) v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w).$$

Finally, we consider the measurable space $(\Omega, \mathscr{A})$, where $\Omega = \Theta \times \Theta_T$ and $\mathscr{A}$ is the completion of the σ-field $\mathscr{B}(\Theta) \otimes \mathscr{B}(\Theta_T)$ by the collection $\mathscr{N}$ of null sets under the probability measure

$$\mathbf{P}(G \times F_1 \times F_2) = \int_G Q_1(x,w;F_1)Q_2(x,w;F_2)v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w); \quad F_1, F_2 \in \mathscr{B}(\Theta_T);$$

$$G \in \mathscr{B}(\mathbf{R}) \otimes \mathscr{B}(\mathbf{C}([0,T],\mathbf{R}^Q)).$$

Moreover, we define the process $(w_t, x_t^{(1)}, y_t^{(1)}, z_t^{(1)}, x_t^{(2)}, y_t^{(2)}, z_t^{(2)})_{0 \leqslant t \leqslant T}$ on the space Ω by letting

$$w, x^{(1)}, x^{(2)} : (t, h^{(P)}) \in [0,T] \times \mathbf{C}([0,T],\mathbf{R}^P) \mapsto h^{(P)}(t),$$

$$y^{(1)}, y^{(2)} : (t, h^{(Q)}) \in [0,T] \times \mathbf{C}([0,T],\mathbf{R}^Q) \mapsto h^{(Q)}(t),$$

$$z^{(1)}, z^{(2)} : (t, h^{(Q \times P)}) \in [0,T] \times L^2([0,T],\mathbf{R}^{Q \times P}) \mapsto \lim_{n \to \infty} n \int_{(t-1/n)^+}^t h^{(Q \times P)}(s)\,\mathrm{d}s,$$

$$\text{if } n \int_{(t-1/n)^+}^t h^{(Q \times P)}(s)\,\mathrm{d}s \text{ converges as } n \to +\infty \text{ and } 0 \text{ otherwise,}$$

and we define the random variable $x : u \in \mathbf{R}^{Q \times P} \mapsto u$.

By the way, note that for every $h \in L^2([0,T],\mathbf{R}^{Q \times P}), z^{(1)}(h)$ and $z^{(2)}(h)$ are correctly defined i.e. they do not depend on the choice of the version of h. Moreover, thanks to the fundamental Lebesgue's Theorem, $z^{(1)}(h)$ (resp. $z^{(2)}(h)$) is also a version of h.

Let us now endow $(\Omega, \mathscr{A}, \mathbf{P})$ with the filtration $\{\mathscr{G}_t\}_{0 \leqslant t \leqslant T}$, where

$$\forall t \in [0,T], \quad \mathscr{G}_t = \tilde{\mathscr{H}}_{t+}; \qquad \tilde{\mathscr{H}}_t = \sigma\{\mathscr{N} \cup \mathscr{H}_t\},$$

with the following notations:

$$\forall t \in [0,T], \quad \mathscr{H}_t = \sigma\{\Pi_t\},$$

where

$$\Pi_t : \mathbf{R}^P \times \Gamma_T \to \mathbf{R}^P \times \Gamma_t, \quad (x,h) \mapsto (x, h_{[0,t]}),$$

and $\forall\ 0 \leqslant s \leqslant T$,

$$\Gamma_s = \mathbf{C}([0,s],\mathbf{R}^P) \times (\mathbf{C}([0,s],\mathbf{R}^P) \times \mathbf{C}([0,s],\mathbf{R}^Q) \times L^2([0,s],\mathbf{R}^{Q\times P}))^2.$$

Moreover, in the definition of $\mathscr{H}_t$, $\mathbf{R}^P \times \Gamma_t$ is endowed with its borelian σ-field.

We then prove that x is $\mathscr{G}_0$-measurable and that $(w_t, x_t^{(1)}, y_t^{(1)}, z_t^{(1)}, x_t^{(2)}, y_t^{(2)}, z_t^{(2)})_{0\leqslant t\leqslant T}$ is $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-progressively measurable. Moreover, for $i \in \{1,2\}$,

$$\mathbf{P}\{\omega \in \Omega; (x, w, x^{(i)}, y^{(i)}, z^{(i)}) \in A\} = v_i\{(\xi^i, W^{(i)}, X^{(i)}, Y^{(i)}, Z^{(i)}) \in A\}; \quad A \in \mathscr{B}(\Theta).$$

Let us assume for the moment that $(w_t)_{0\leqslant t\leqslant T}$ is a $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$ Brownian motion. Then, approximating the process $(z_t^{(1)})_{0\leqslant t\leqslant T}$ (resp. $(z_t^{(2)})_{0\leqslant t\leqslant T}$) with a sequence of simple processes, we prove that $\mathbf{P}$-a.s.:

$$\begin{cases} \forall t \in [0,T], \\ x_t^{(i)} = x + \displaystyle\int_0^t f(s, x_s^{(i)}, y_s^{(i)}, z_s^{(i)})\,\mathrm{d}s + \int_0^t \sigma(s, x_s^{(i)}, y_s^{(i)})\,\mathrm{d}w_s, \\ y_t^{(i)} = h(x_T^{(i)}) + \displaystyle\int_t^T g(s, x_s^{(i)}, y_s^{(i)}, z_s^{(i)})\,\mathrm{d}s - \int_t^T z_s^{(i)}\,\mathrm{d}w_s, \\ \mathbf{E}\displaystyle\int_0^T (|x_t^{(i)}|^2 + |y_t^{(i)}|^2 + |z_t^{(i)}|^2)\,\mathrm{d}t < \infty. \end{cases}$$

Using Theorem 1.1, this shows that

$$y_0^{(1)} = y_0^{(2)},$$

and therefore proves that the map θ only depends on the coefficients (f, g, h, σ).

Actually, we just have to prove that $(w_t)_{0\leqslant t\leqslant T}$ is a $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$ Brownian motion: we follow the proof given in Rogers and Williams (1987) in the SDE case. Let us firstly define

$$\pi_t : \mathbf{C}([0,T],\mathbf{R}^P) \to \mathbf{C}([0,T],\mathbf{R}^P), \quad h \mapsto h_{|[0,t]}; \quad \pi_t' : \gamma_T \to \gamma_t, \ \ h \mapsto h_{|[0,t]},$$

where $\forall\ 0 \leqslant s \leqslant T$,

$$\gamma_s = \mathbf{C}([0,s],\mathbf{R}^P) \times \mathbf{C}([0,s],\mathbf{R}^Q) \times L^2([0,s],\mathbf{R}^{Q\times P}).$$

Endowing $\mathbf{C}([0,t],\mathbf{R}^P)$ and γ_t with their borelian σ-field, we define

$$\mathscr{K}_t = \sigma\{\pi_t\}; \qquad \mathscr{K}_t' = \sigma\{\pi_t'\}.$$

Using the separability of the spaces $\mathbf{C}([0,t],\mathbf{R}^P)$ and γ_t, we see that

$$\mathscr{K}_t = \sigma\{w_r,\ r \leqslant t\},$$

and that $\forall i \in \{1,2\}$, $\forall A \in \mathscr{K}_t'$, the set $\{(X^{(i)}, Y^{(i)}, Z^{(i)}) \in A\}$ belongs to $\mathscr{G}_t^{(i)}$.

Now, considering $A \in \mathscr{K}'_t$, we want to show that, for $i \in \{1,2\}$, the map

$$\mathbf{R}^P \times \mathbf{C}([0,T],\mathbf{R}^P) \to [0,1], \quad (x,w) \mapsto Q_i(x,w;A)$$

is measurable with respect to the completion of the σ-field $\mathscr{B}(\mathbf{R}^P) \otimes \mathscr{K}_t$ under the probability measure $v \otimes \mathbf{P}_*$, denoted $\overline{\mathscr{B}(\mathbf{R}^P) \otimes \mathscr{K}_t}$.

Indeed, let us consider $F \in \mathscr{B}(\mathbf{R}^P)$, $G_1 \in \mathscr{K}_t$, and $G_2 \in \sigma\{w_r - w_t,\ r \geqslant t\}$. Then, $\forall i \in \{1,2\}$,

$$\begin{aligned}
&\int \mathbf{1}_F(x)\mathbf{1}_{G_1}(w)\mathbf{1}_{G_2}(w)Q_i(x,w;A)v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w)\\
&\quad = \mathscr{E}_i(\mathbf{1}_F(X_0)\mathbf{1}_{G_1}(W^{(i)})\mathbf{1}_{G_2}(W^{(i)})\mathbf{1}_A(X^{(i)},Y^{(i)},Z^{(i)}))\\
&\quad = \mathscr{E}_i(\mathbf{1}_F(X_0)\mathbf{1}_{G_1}(W^{(i)})\mathbf{1}_A(X^{(i)},Y^{(i)},Z^{(i)}))\mathscr{E}_i(\mathbf{1}_{G_2}(W^{(i)}))\\
&\quad = \int \mathbf{1}_F(x)\mathbf{1}_{G_1}(w)Q_i(x,w;A)v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w)\int \mathbf{1}_{G_2}(w)v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w).
\end{aligned}$$

Hence, using Exercise (17.10) Chapter V of Rogers and Williams (1987), this proves that, for $i \in \{1,2\}$, the map $(x,w) \mapsto Q_i(x,w;A)$ is measurable with respect to $\overline{\mathscr{B}(\mathbf{R}^P) \otimes \mathscr{K}_t}$.

Let us now prove that the process $(w_t)_{0 \leqslant t \leqslant T}$ is a $\{\mathscr{G}_t\}_{0 \leqslant t \leqslant T}$ Brownian motion. Let us consider $(A,A') \in (\mathscr{K}'_t)^2, F \in \mathscr{B}(\mathbf{R}^P), G_1 \in \mathscr{K}_t$, and $G_2 \in \sigma\{w_r - w_t, r \geqslant t\}$. Then,

$$\begin{aligned}
&\mathbf{E}(\mathbf{1}_F(x)\mathbf{1}_{G_1}(w)\mathbf{1}_{G_2}(w)\mathbf{1}_A(x^{(1)},y^{(1)},z^{(1)})\mathbf{1}_{A'}(x^{(2)},y^{(2)},z^{(2)}))\\
&\quad = \int \mathbf{1}_F(x)\mathbf{1}_{G_1}(w)\mathbf{1}_{G_2}(w)Q_1(x,w;A)Q_2(x,w;A')v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w)\\
&\quad = \int \mathbf{1}_F(x)\mathbf{1}_{G_1}(w)Q_1(x,w;A)Q_2(x,w;A')v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w)\int \mathbf{1}_{G_2}(w)v(\mathrm{d}x)\mathbf{P}_*(\mathrm{d}w)\\
&\quad = \mathbf{E}(\mathbf{1}_F(x)\mathbf{1}_{G_1}(w)\mathbf{1}_A(x^{(1)},y^{(1)},z^{(1)})\mathbf{1}_{A'}(x^{(2)},y^{(2)},z^{(2)}))\mathbf{E}(\mathbf{1}_{G_2}(w)).
\end{aligned}$$

Noting that $\mathscr{H}_t = \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{K}_t \otimes \mathscr{K}'_t \otimes \mathscr{K}'_t$, we conclude that $(w_t)_{0 \leqslant t \leqslant T}$ is a $\{\mathscr{G}_t\}_{0 \leqslant t \leqslant T}$ Brownian motion. $\square$

Corollary 1.7 (Dependence upon the coefficients). *Suppose that* (A1) *is in force and* $T \leqslant C_K^{(2)}$, *and keep the notations of Corollary* 1.5. *Let* $(f_n, g_n, h_n, \sigma_n)_{n \in \mathbf{N}}$ *be a sequence of functions satisfying Assumption* (A1) *with respect to the same constants* K *and* Λ *as* (f,g,h,σ), *and verifying*

For a.e. $t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q \times P}$, $(f_n,g_n,h_n,\sigma_n)(t,x,y,z) \to (f,g,h,\sigma)(t,x,y,z)$, *as* $n \to +\infty$.

If for every $\mathscr{G}_0$*-measurable random vector* ξ *with finite second moment*, $(X_t^{n,0,\xi}, Y_t^{n,0,\xi}, Z_t^{n,0,\xi})_{0\leqslant t\leqslant T}$ *stands for the solution of*

$$\begin{cases} \forall t \in [0,T], \\ X_t = \xi + \displaystyle\int_0^t f_n(s,X_s,Y_s,Z_s)\,\mathrm{d}s + \int_0^t \sigma_n(s,X_s,Y_s)\,\mathrm{d}B_s, \\ Y_t = h_n(X_T) + \displaystyle\int_t^T g_n(s,X_s,Y_s,Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\displaystyle\int_0^T (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty, \end{cases}$$

then, *as* $n \to +\infty$,

$$\mathbf{E}\sup_{0\leqslant s\leqslant T} |X_s^{n,0,\xi} - X_s^{0,\xi}|^2 + \mathbf{E}\sup_{0\leqslant s\leqslant T} |Y_s^{n,0,\xi} - Y_s^{0,\xi}|^2 + \mathbf{E}\int_0^T |Z_s^{n,0,\xi} - Z_s^{0,\xi}|^2\,\mathrm{d}s \to 0. \tag{1.7.1}$$

In particular, *as* $n \to +\infty$,

$$\theta_n \to \theta, \tag{1.7.2}$$

uniformly on every compact set of $[0,T]\times\mathbf{R}^P$, *where* θ_n *stands for the map associated by means of Corollary* 1.5 *to the coefficients* f_n, g_n, h_n, *and* σ_n.

Proof. Let T be in $[0, C_K^{(2)}]$. Then, applying Theorem 1.3 as well as Lebesgue's convergence Theorem, we prove (1.7.1). In particular, we deduce that:

$$\forall (t,x) \in [0,T]\times\mathbf{R}^P, \quad \theta_n(t,x) \to \theta(t,x).$$

In fact, property (1.5.2) proves that the maps $(\theta_n)_{n\in\mathbf{N}}$ are equicontinuous on every compact set of $[0,T]\times\mathbf{R}^P$. This shows that the convergence is uniform on every compact set.

This completes the proof of Corollary 1.7. □

Some comments on the sequel of the paper. The property (1.5.3) is certainly one of the main results of the first section, because it permits us to describe locally, that means on a neighborhood of the bound T, any solution of the problem (**E**) over an arbitrarily prescribed time duration T. In particular, the role played by the map $\theta(T-\delta,.)$ at time $T-\delta$ for a small enough δ exactly matches the role played by the map h at time T. Hence, we can hope to describe, by means of a running-down induction, every solution on its whole interval of definition, whatever the time duration may be.

Nevertheless, such a kind of construction requires an efficient control of the neighborhood on which the description is possible. Under Assumption (A1), this control is based on K: the bigger K is, the smaller the description neighborhood is. So, if the Lispchitz constant of the map $\theta(t,.)$ grows during the latter induction, the length of the description neighborhood decreases, and it is then not clear that the solution can be extended to the whole interval $[0,T]$: the Lipschitz constant may explode in a finite time and in such a case the description remains local. Actually, Assumption (A1) is too weak to prevent such a behavior.

However, we know (see for example Ma et al., 1994, and more recently Hu and Yong, 2000) that in the case of regular coefficients, the control of the Lispchitz constant of the maps $\theta(t,.)$ corresponds to a control of the gradient of the solution of a parabolic quasi-linear system of PDEs (c.f. Introduction): under appropriate assumptions, the problem (E′) is solvable and the gradient of its solution is bounded in such a way that the control of the description neighborhood is efficient. The strategy in the next part follows from this remark and aims to keep, under appropriate assumptions and along a sequence of regularized coefficients whose associated system of PDEs is solvable and satisfies the latter gradient boundedness property, this control of the neighborhood of description.

2. Application to a non-degenerate diffusion coefficient case

We need now new assumptions on the coefficients given by the following:

Assumption (A2). For a non-negative real T, we say that the functions f,g,h and σ satisfy Assumption (A2) if there exist four constants K,k,Λ and $\lambda>0$, such that they satisfy both Assumption (A1) with respect to the constants K and Λ and the following properties:

(A2.1): For every $t\in[0,T]$, for every $(x,y)\in\mathbf{R}^P\times\mathbf{R}^Q$ and for every $(x',y')\in\mathbf{R}^P\times\mathbf{R}^Q$,

$$|\sigma(t,x,y)-\sigma(t,x',y')|^2\leqslant k^2(|x-x'|^2+|y-y'|^2),$$

$$|h(x)-h(x')|\leqslant k|x-x'|.$$

This means that we have, for the convenience of our demonstration, to distinguish the Lipschitz constants of σ and h from the constant K.

(A2.2): For every $t\in[0,T]$, for every $(x,y,z)\in\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z)|\leqslant\Lambda(1+|y|+|z|),$$

$$|g(t,x,y,z)|\leqslant\Lambda(1+|y|+|z|),$$

$$|\sigma(t,x,y)|\leqslant\Lambda(1+|y|),$$

$$|h(x)|\leqslant\Lambda.$$

(A2.3): For every $(t,x,y)\in[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q$,

$$\forall\zeta\in\mathbf{R}^P,\quad\langle\zeta,a(t,x,y)\zeta\rangle\geqslant\lambda|\zeta|^2,$$

where the function a is defined as follows on $[0,T] \times \mathbf{R}^P \times \mathbf{R}^Q$,

$$\forall (t,x,y) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q, \quad a(t,x,y) = \sigma\sigma^*(t,x,y).$$

(A2.4): The function σ is continuous on its definition set.

The following lemma is essential for the sequel of the paper:

Lemma 2.1. *Assume that $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ and $\tilde{\sigma}$ are bounded $\mathbf{C}^\infty$ functions with bounded derivatives of every order and satisfy Assumption* (A2) *with respect to the constants K, k, Λ, and λ. Then, setting $\tilde{a} = \tilde{\sigma}\tilde{\sigma}^*$, the following system of PDEs*:

$$\begin{cases} \forall (t,x) \in [0,T] \times \mathbf{R}^P, \ \forall \ell \in \{1,..,Q\}, \\ \dfrac{\partial \tilde{\theta}_\ell}{\partial t}(t,x) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{P} \tilde{a}_{i,j}(t,x,\tilde{\theta}(t,x)) \dfrac{\partial^2 \tilde{\theta}_\ell}{\partial x_i \partial x_j}(t,x) \\ + \displaystyle\sum_{i=1}^{P} \tilde{f}_i(t,x,\tilde{\theta}(t,x), \nabla_x \tilde{\theta}(t,x) \tilde{\sigma}(t,x,\tilde{\theta}(t,x))) \dfrac{\partial \tilde{\theta}_\ell}{\partial x_i}(t,x) \\ + \tilde{g}_\ell(t,x,\tilde{\theta}(t,x), \nabla_x \tilde{\theta}(t,x) \tilde{\sigma}(t,x,\tilde{\theta}(t,x))) = 0, \\ \forall x \in \mathbf{R}^P, \quad \tilde{\theta}(T,x) = \tilde{h}(x), \end{cases} \tag{2.1.1}$$

admits a unique bounded solution $\tilde{\theta} \in \mathbf{C}^{1,2}([0,T] \times \mathbf{R}^P, \mathbf{R}^Q)$. It satisfies

$$\forall (i,j) \in \{1,..,P\}^2, \frac{\partial \tilde{\theta}}{\partial x_i} \text{ and } \frac{\partial^2 \tilde{\theta}}{\partial x_i \, \partial x_j} \text{ are bounded on } \mathbf{R}^P. \tag{2.1.2}$$

In addition, there exist a constant $\tilde{C}$, only depending on Λ and T, and two constants $\tilde{\Gamma}$ and $\tilde{\kappa}$, only depending on k, λ, Λ, P, Q and T, such that

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^P} |\tilde{\theta}(t,x)| \leqslant \tilde{C}, \tag{2.1.3}$$

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^P} |\nabla_x \tilde{\theta}(t,x)| \leqslant \tilde{\Gamma}, \tag{2.1.4}$$

$$\forall (t,t') \in [0,T]^2, \ \forall x \in \mathbf{R}^P, \quad |\tilde{\theta}(t',x) - \tilde{\theta}(t,x)| \leqslant \tilde{\kappa} |t'-t|^{1/2}. \tag{2.1.5}$$

Moreover, for every $t \in [0,T]$, for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the SDE:

$$\begin{aligned} \forall t \leqslant s \leqslant T, \quad \tilde{U}_s = \xi &+ \int_t^s \tilde{f}(r, \tilde{U}_r, \tilde{\theta}(r, \tilde{U}_r), \nabla_x \tilde{\theta}(r, \tilde{U}_r) \tilde{\sigma}(r, \tilde{U}_r, \tilde{\theta}(r, \tilde{U}_r))) \, \mathrm{d}r \\ &+ \int_t^s \tilde{\sigma}(r, \tilde{U}_r, \tilde{\theta}(r, \tilde{U}_r)) \, \mathrm{d}B_r, \end{aligned}$$

admits a unique solution, denoted by $(\tilde{U}_s^{t,\xi})_{t \leqslant s \leqslant T}$, and the process $(\tilde{U}_s^{t,\xi}, \tilde{V}_s^{t,\xi}, \tilde{W}_s^{t,\xi})_{t \leqslant s \leqslant T}$, given by

$$\forall t \leqslant s \leqslant T, \quad \tilde{V}_s^{t,\xi} = \tilde{\theta}(s, \tilde{U}_s^{t,\xi}), \quad \tilde{W}_s^{t,\xi} = \nabla_x \tilde{\theta}(s, \tilde{U}_s^{t,\xi}) \tilde{\sigma}(s, \tilde{U}_s^{t,\xi}, \tilde{V}_s^{t,\xi}), \tag{2.1.6}$$

satisfies the FBSDE associated to $(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\sigma})$ and to the initial condition (t,ξ).

Proof. Thanks to Proposition 3.3 of Ma et al. (1994) (see also Theorem 7.1 Chapter VII of Ladyzenskaja et al., 1968), we know that the system (2.1.1) admits a unique bounded classical solution denoted $\tilde{\theta}$ (note that we give a more probabilistic proof of this result in Appendix B). Still from Ma et al. and Ladyzenskaja et al., we know that $\tilde{\theta}$ satisfies (2.1.2).

Let us show by means of probabilistic tools that (2.1.3) holds. To this end, let us define for every $(t,x)\in[0,T]\times\mathbf{R}^P$:

$$\tilde{F}(t,x)=\tilde{f}(t,x,\tilde{\theta}(t,x),\nabla_x\tilde{\theta}(t,x)\sigma(t,x,\tilde{\theta}(t,x))),$$

$$\tilde{\Sigma}(t,x)=\tilde{\sigma}(t,x,\tilde{\theta}(t,x)).$$

For every $t\in[0,T]$, for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the SDE

$$\tilde{U}_s^{t,\xi}=\xi+\int_t^s\tilde{F}(r,\tilde{U}_r^{t,\xi})\,\mathrm{d}r+\int_t^s\tilde{\Sigma}(r,\tilde{U}_r^{t,\xi})\,\mathrm{d}B_r,$$

admits a unique solution. We then define $\forall t\leqslant s\leqslant T$,

$$\tilde{V}_s^{t,\xi}=\tilde{\theta}(s,\tilde{U}_s^{t,\xi}),\quad \tilde{W}_s^{t,\xi}=\nabla_x\tilde{\theta}(s,\tilde{U}_s^{t,\xi})\Sigma(s,\tilde{U}_s^{t,\xi}).$$

Therefore, Itô's formula and system (2.1.1) show that $\forall t\leqslant s\leqslant T$,

$$\tilde{V}_s^{t,\xi}=\tilde{h}(\tilde{U}_T^{t,\xi})+\int_s^T\tilde{g}(r,\tilde{U}_r^{t,\xi},\tilde{V}_r^{t,\xi},\tilde{W}_r^{t,\xi})\,\mathrm{d}r-\int_s^T\tilde{W}_r^{t,\xi}\,\mathrm{d}B_r.$$

Hence, the process $(\tilde{U}^{t,\xi},\tilde{V}^{t,\xi},\tilde{W}^{t,\xi})$ is a solution of the FBSDE associated to the coefficients $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ and $\tilde{\sigma}$ and to the initial condition (t,ξ).

Moreover, fixing arbitrarily $c\in\mathbf{R}$ and applying Itô's formula to the semimartingale $(\mathrm{e}^{cs}|\tilde{V}_s^{t,x}|^2)_{t\leqslant s\leqslant T}$, we deduce that for every $t\leqslant s\leqslant T$ and for every $x\in\mathbf{R}^P$:

$$\begin{aligned}
&\mathrm{e}^{cs}|\tilde{V}_s^{t,x}|^2+\int_s^T\mathrm{e}^{cr}|\tilde{W}_r^{t,x}|^2\,\mathrm{d}r\\
&\quad\leqslant \mathrm{e}^{cT}|\tilde{V}_T^{t,x}|^2+\int_s^T\mathrm{e}^{cr}[2\Lambda(1+|\tilde{V}_r^{t,x}|+|\tilde{W}_r^{t,x}|)|\tilde{V}_r^{t,x}|-c|\tilde{V}_r^{t,x}|^2]\,\mathrm{d}r\\
&\qquad-2\int_s^T\mathrm{e}^{cr}\langle\tilde{V}_r^{t,x},\tilde{W}_r^{t,x}\,\mathrm{d}B_r\rangle\\
&\quad\leqslant \mathrm{e}^{cT}|\tilde{V}_T^{t,x}|^2+\int_s^T\mathrm{e}^{cr}[\Lambda+(3\Lambda+2\Lambda^2-c)|\tilde{V}_r^{t,x}|^2+\frac{1}{2}|\tilde{W}_r^{t,x}|^2]\,\mathrm{d}r\\
&\qquad-2\int_s^T\mathrm{e}^{cr}\langle\tilde{V}_r^{t,x},\tilde{W}_r^{t,x}\,\mathrm{d}B_r\rangle.
\end{aligned}$$

Choosing $c=3\Lambda+2\Lambda^2$, and taking the conditional expectation given $\mathscr{G}_t$, we deduce that there exists a constant $\tilde{C}$, only depending on Λ and T, such that

$$\forall(t,x)\in[0,T]\times\mathbf{R}^P,\quad|\tilde{\theta}(t,x)|\leqslant\tilde{C}. \tag{2.1.7}$$

Then, using Theorem 6.1 Chapter VII of Ladyzenskaja et al. (1968), we can estimate the supremum norm of $|\nabla_x \tilde{\theta}|^2$ on every compact set of $[0,T] \times \mathbf{R}^P$. Indeed, for every $n \in \mathbf{N}^*$, we can apply this theorem to the cylinders $[0,T] \times \{x \in \mathbf{R}^P, |x| \leqslant n\}$ and $[0,T] \times \{x \in \mathbf{R}^P, |x| \leqslant n+1\}$. In particular, the quantity $\sup_{\{t \in [0,T], |x| \leqslant n\}} |\nabla_x \tilde{\theta}(t,x)|^2$ is estimated in terms of $\tilde{C}$, k, λ, Λ, P and Q, the distance between $\{x \in \mathbf{R}^P, |x| \leqslant n\}$ and $\partial\{x \in \mathbf{R}^P, |x| \leqslant n+1\}$ being equal to 1.

In particular, there exists a constant $\tilde{\Gamma}$, only depending on k, Λ, λ, P, Q and T such that

$$\forall (t,x) \in [0,T] \times \mathbf{R}^P, \quad |\nabla_x \tilde{\theta}(t,x)| \leqslant \tilde{\Gamma}. \tag{2.1.8}$$

Lastly, let us prove (2.1.5). Let us now consider $0 \leqslant u \leqslant v \leqslant T$. Then, using (2.1.7) and (2.1.8), we show that there exists a constant $\tilde{\kappa}$ only depending on k, Λ, λ, P, Q and T, such that

$$\mathbf{E}|\tilde{V}_v^{u,x} - \tilde{V}_u^{u,x}|^2 \leqslant \tilde{\kappa}(v-u), \quad \mathbf{E}|\tilde{U}_v^{u,x} - \tilde{U}_u^{u,x}|^2 \leqslant \tilde{\kappa}(v-u).$$

Hence, by modifying $\tilde{\kappa}$ if necessary, and using $\tilde{V}_v^{u,x} = \tilde{\theta}(v, \tilde{U}_v^{u,x})$,

$$\begin{aligned} |\tilde{\theta}(u,x) - \tilde{\theta}(v,x)|^2 &\leqslant 2[\mathbf{E}|\tilde{\theta}(u,x) - \tilde{V}_v^{u,x}|^2 + \mathbf{E}|\tilde{V}_v^{u,x} - \tilde{\theta}(v,x)|^2] \\ &\leqslant \tilde{\kappa}[(v-u) + \mathbf{E}|\tilde{U}_v^{u,x} - x|^2] \leqslant \tilde{\kappa}(v-u). \end{aligned}$$

This shows (2.1.5). $\square$

Proposition 2.2. *Under Assumption* (A2), *there exists a sequence of* $\mathbf{C}^\infty$ *functions* $(f_n, g_n, h_n, \sigma_n)_{n \in \mathbf{N}^*}$ *satisfying for every* $n \in \mathbf{N}^*$ *Assumption* (A2) *with respect to the constants* $K + 4\Lambda$, k, 2Λ *and* $\lambda/2$, *and such that*

For a.e. $t \in [0,T], \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q \times P}$, $(f_n, g_n, h_n, \sigma_n)(t,x,y,z) \to (f,g,h,\sigma)(t,x,y,z)$, *as* $n \to +\infty$. *Moreover, for every* $n \in \mathbf{N}^*$, *letting* $a_n = \sigma_n \sigma_n^*$, *the following system of PDEs*:

$$\begin{cases} \forall (t,x) \in [0,T] \times \mathbf{R}^P, \forall \ell \in \{1,..,Q\}, \\ \dfrac{\partial (\theta_n)_\ell}{\partial t}(t,x) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^P (a_n)_{i,j}(t,x,\theta_n(t,x)) \dfrac{\partial^2 (\theta_n)_\ell}{\partial x_i \, \partial x_j}(t,x) \\ + \displaystyle\sum_{i=1}^P (f_n)_i(t,x,\theta_n(t,x), \nabla_x \theta_n(t,x) \sigma_n(t,x,\theta_n(t,x))) \dfrac{\partial (\theta_n)_\ell}{\partial x_i}(t,x) \\ + (g_n)_\ell(t,x,\theta_n(t,x), \nabla_x \theta_n(t,x) \sigma_n(t,x,\theta_n(t,x))) = 0, \\ \forall x \in \mathbf{R}^P, \quad \theta_n(T,x) = h_n(x); \end{cases} \tag{2.2.1}$$

admits a unique bounded solution $\theta_n \in \mathbf{C}^{1,2}([0,T] \times \mathbf{R}^P, \mathbf{R}^Q)$. *This one satisfies* (2.1.2).

In addition, there exist a constant C, *only depending on* Λ *and* T, *and two constants* Γ *and* κ, *only depending on* k, Λ, λ, P, Q *and* T, *such that* $\forall n \in \mathbf{N}^*$,

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^P} |\theta_n(t,x)| \leqslant C, \tag{2.2.2}$$

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^P} |\nabla_x \theta_n(t,x)| \leqslant \Gamma, \tag{2.2.3}$$

$$\forall (t,t')\in[0,T]^2,\ \forall x\in\mathbf{R}^P, \quad |\theta_n(t',x)-\theta_n(t,x)| \leqslant \kappa|t'-t|^{1/2}. \tag{2.2.4}$$

Proof. In order to apply Lemma 2.1 to a good regularization sequence of (f,g,h,σ), we introduce the following objects:

(i) $(\rho_n)_{n\in\mathbf{N}}, (\rho_n^1)_{n\in\mathbf{N}}, (\rho_n^2)_{n\in\mathbf{N}}$, and $(\rho_n^3)_{n\in\mathbf{N}}$ are four mollifiers on (respectively) $\mathbf{R}$, $\mathbf{R}^P$, $\mathbf{R}^Q$ and $\mathbf{R}^{Q\times P}$ defined by

$$\rho_n(.) = cn\varphi(n|.|); \quad \rho_n^1(.) = c_1 n^P \varphi(n|.|);$$

$$\rho_n^2(.) = c_2 n^Q \varphi(n|.|); \quad \rho_n^3(.) = c_3 n^{Q\times P} \varphi(n|.|),$$

where $\forall x\in\mathbf{R}, \varphi(x)=\exp(-1/(x^2-1))\mathbf{1}_{]-1,1[}(x)$, and $c,\ c_1,\ c_2$ and c_3 are four constants of normalization.

(ii) For every $N\in\mathbf{N}^*$ and for every $r>0$, we define the following map:

$$\tau_r^{(N)} : \mathbf{R}^N \to \mathbf{R}^N, \quad x \mapsto \frac{r}{r\vee|x|}\,x,$$

which is 1-lispchitzian and satisfies $\forall x\in\mathbf{R}^N, |\tau_r^{(N)}(x)| \leqslant r\wedge|x|$. We set for every $r>0$, $\tau_r^1=\tau_r^{(P)}$, $\tau_r^2=\tau_r^{(Q)}$ and $\tau_r^3=\tau_r^{(Q\times P)}$.

Moreover, we also let

$$\pi_r : \mathbf{R}^+ \to \mathbf{R}^+, \quad x\mapsto \mathbf{1}_{[0,r]}(x) + \frac{2r-x}{r}\mathbf{1}_{[r,2r]}(x),$$

which is $(1/n)$-lispchitzian and satisfies $\forall x\in\mathbf{R}^+,\ 0\leqslant \pi_r(x)\leqslant 1$.

(iii) We extend the function (f,g,σ) to $\mathbf{R}\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$ by putting

$$\forall (t,x,y,z)\in\mathbf{R}\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P},$$

$$(f,g,\sigma)(t,x,y,z) = (f,g,\sigma)(0,x,y,z) \quad \text{if } t<0,$$

$$(f,g,\sigma)(t,x,y,z) = (f,g,\sigma)(T,x,y,z) \quad \text{if } t>T.$$

(iv) $\forall n\in\mathbf{N}^*$, we denote by ω_n the modulus of continuity of σ on the compact set $[0,T]\times\{x\in\mathbf{R}^P, |x|\leqslant n\}\times\{y\in\mathbf{R}^Q, |y|\leqslant n\}$. Therefore, for every $n\in\mathbf{N}^*$, there exists an integer $p_n\geqslant 2$ such that

$$\sup_{|x|\leqslant n, |y|\leqslant n} |\sigma(t,x,y)|\omega_n\left(\frac{4}{p_n}\right) \leqslant \frac{\lambda}{2n},$$

where $(p_n)_{n\in\mathbf{N}}$ is chosen strictly increasing and growing up to $+\infty$.

(v) Lastly, for every $n\in\mathbf{N}^*$, we define

$$f_n(t,x,y,z) = \int f(t-s, x-u, \tau_n^2(y-v), \tau_n^3(z-w))$$

$$\rho_{p_n}(s)\rho_{p_n}^1(u)\rho_{p_n}^2(v)\rho_{p_n}^3(w)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w,$$

$$g_n(t,x,y,z) = \int \pi_n(|y-v|)g(t-s,x-u,y-v,\tau_n^3(z-w))$$
$$\rho_{p_n}(s)\rho^1_{p_n}(u)\rho^2_{p_n}(v)\rho^3_{p_n}(w)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w,$$

$$h_n(x) = \int h(x-u)\rho^1_{p_n}(u)\,\mathrm{d}u$$

$$\sigma_n(t,x,y) = \int \sigma(t-s,\tau_n^1(x-u),\tau_n^2(y-v))\rho_{p_n}(s)\rho^1_{p_n}(u)\rho^2_{p_n}(v)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v.$$

Such choices will be justified at the end of the demonstration of Proposition 2.2.

Let us assume for the moment that for every $n\in\mathbf{N}^*$, f_n, g_n, h_n and σ_n satisfy (the proof will be given at the end of the demonstration of Proposition 2.2):

(A.n2.1): $\forall n\in\mathbf{N}^*$, f_n, g_n, h_n and σ_n satisfy Assumption (A2) with respect to the constants $K+4\Lambda$, k, 2Λ, and $\lambda/2$.

(A.n2.2): For every $n\in\mathbf{N}^*$, the functions f_n, g_n, h_n and σ_n are of class $\mathbf{C}^\infty$ on their definition set. Moreover, for every $n\in\mathbf{N}^*$, the functions f_n, g_n, h_n and σ_n as well as their derivatives of every order are bounded.

(A.n2.3): For a.e. $t\in[0,T], \forall(x,y,z)\in\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$, $(f_n,g_n,h_n,\sigma_n)(t,x,y,z)\to(f,g,h,\sigma)(t,x,y,z)$, as $n\to+\infty$.

We denote by (A.n2) the set of Assumptions (A.n2.1)–(A.n2.3).

Therefore, from Lemma 2.1, we know that for every $n\in\mathbf{N}^*$, the system of partial differential equations (2.2.1) admits a unique bounded classical solution denoted θ_n. For every $n\in\mathbf{N}^*, \theta_n$ satisfies (2.1.2). Still from Lemma 2.1, there exist a constant C, only depending on Λ and T, and two constants Γ and κ, only depending on k, Λ, λ, P, Q and T, such that (2.2.2), (2.2.3) and (2.2.4) hold for every $n\in\mathbf{N}^*$.

Let us now prove that (A.n2) holds. Let us firstly show (A.n2.1). Thanks to (A2) and to (ii), we deduce that for every $n\in\mathbf{N}^*$ and for every $(t,x,y,z)\in[0,T]$:

$$|f_n(t,x,y,z)| \leqslant \Lambda\left(1+\left(|y|+\frac{1}{p_n}\right)+\left(|z|+\frac{1}{p_n}\right)\right).$$

Hence, for every $n\in\mathbf{N}^*, f_n$ satisfies the first line of (A2.2) with respect to 2Λ. In the same way, we prove that g_n, σ_n and h_n satisfy the second, third and fourth lines of (A2.2) with respect to 2Λ, 2Λ and Λ.

Moreover, thanks to (A2.1) and to (ii), we deduce that for every $n\in\mathbf{N}^*$, h_n and σ_n satisfy (A2.1) with respect to k. In the same way, we deduce from (A1.1), (A1.2) and (ii) that for every $n\in\mathbf{N}^*, f_n$ satisfies the first lines of (A1.1) and (A1.2) with respect to K, and that g_n satisfies the second line of (A1.1) with respect to K.

In addition, note from (ii) and (A1.2) that for every $n\in\mathbf{N}^*$ and for every $(t,x,y,y',z)\in[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$:

$$\begin{aligned}\langle y'-y, g_n(t,x,y',z)-g_n(t,x,y,z)\rangle &\leqslant K|y'-y|^2+\frac{1}{n}\Lambda(1+2n+n)|y'-y|^2\\ &\leqslant (4\Lambda+K)|y'-y|^2.\end{aligned}$$

By the way, due to (A1.2), note that the way we have regularized the coefficient f cannot be applied to the case of g (we recall that we want g_n and its derivatives to be bounded).

Finally, for every $n \in \mathbf{N}^*$, for every $(t,x,y) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q$ and for every $\xi \in \mathbf{R}^P$:

$$\langle \xi, a_n(t,x,y)\xi \rangle \geqslant \int \langle \xi, a(t-s, \tau_n^1(x-u), \tau_n^2(y-v))\xi \rangle \rho_{p_n}(s)\rho_{p_n}^1(u)\rho_{p_n}^2(v)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v$$

$$-|\xi|^2 \sup_{|x| \leqslant n, |y| \leqslant n} |\sigma(t,x,y)|\omega_n\left(\frac{4}{p_n}\right) \geqslant \left(1 - \frac{1}{2n}\right)\lambda|\xi|^2.$$

This justifies the choice of p_n and completes the proof of (A.n2.1).

(A.n2.2) is readily established. Hence, we now prove that (A.n2.3) holds. Actually, we just have to prove that, as $n \to +\infty$:

$$\text{For a.e. } t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}, \quad f_n(t,x,y,z) \to f(t,x,y,z).$$

To this end, let us firstly define for every $t \in \mathbf{R}$ and for every $p \in \mathbf{N}, \omega_{t,p}$ as the modulus of continuity of $f(t,.)$ on the set $[-p,p]^P \times [-p,p]^Q \times [-p,p]^{Q\times P}$.

From Assumption (A2), for every $(p,m) \in \mathbf{N} \times \mathbf{N}^*$, the map $t \in \mathbf{R} \mapsto \omega_{t,p}(1/m)$ is measurable and bounded. Therefore, it is well known that there exits a set $N \in \mathscr{B}(\mathbf{R})$, such that $\mu(N) = 0$, where μ stands for the Lebesgue measure on $\mathbf{R}$, and

$$\forall t \notin N,\ \forall (p,m) \in \mathbf{N} \times \mathbf{N}^*, \quad \int \omega_{t-s,p}\left(\frac{1}{m}\right)\rho_{p_n}(s)\,\mathrm{d}s \to \omega_{t,p}\left(\frac{1}{m}\right),$$

$$\forall t \notin N,\ \forall (x,y,z) \in \mathbf{Q}^P \times \mathbf{Q}^Q \times \mathbf{Q}^{Q\times P}, \quad \int f(t-s,x,y,z)\rho_{p_n}(s)\,\mathrm{d}s \to f(t,x,y,z),$$

as $n \to +\infty$.

Let us now consider $p \in \mathbf{N}$, $(x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, such that $\max(|x|,|y|,|z|) \leqslant p$, as well as $(x_m,y_m,z_m)_{m\in\mathbf{N}}$ a sequence of $\mathbf{Q}^P \times \mathbf{Q}^Q \times \mathbf{Q}^{Q\times P}$ satisfying

$$\forall m \in \mathbf{N}^*,\ |x-x_m|^2 + |y-y_m|^2 + |z-z_m|^2 \leqslant \frac{1}{m^2}, \quad \text{and}$$

$$\max(|x_m|,|y_m|,|z_m|) \leqslant p.$$

Then, $\forall t \notin N$, $\forall n \geqslant p+1$, $\forall m \in \mathbf{N}^*$,

$$\begin{aligned} &|f_n(t,x,y,z) - f(t,x,y,z)| \\ &\quad\leqslant \left|\int \omega_{t-s,p+1}\left(\frac{2}{p_n}\right)\rho_{p_n}(s)\,\mathrm{d}s\right| + \left|\int \omega_{t-s,p}\left(\frac{1}{m}\right)\rho_{p_n}(s)\,\mathrm{d}s\right| \\ &\qquad + \left|\int f(t-s,x_m,y_m,z_m)\rho_{p_n}(s)\,\mathrm{d}s - f(t,x_m,y_m,z_m)\right| + \left|\omega_{t,p}\left(\frac{1}{m}\right)\right|. \end{aligned}$$

This is enough to conclude that, as $n \to +\infty$:

$$\forall t \notin N,\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}, \quad f_n(t,x,y,z) \to f(t,x,y,z).$$

This concludes the proof of Proposition 2.2. □

Notations. Keeping the notations of Proposition 2.2, let us introduce the following ones:

$$\tilde{K} = \max(k, K + 4\Lambda, \Gamma), \quad \gamma = C^{(2)}_{\tilde{K}}.$$

Corollary 2.3. *Under Assumption* (A2), *and with the latter notations, there exists an integer N given by $N = [T/\gamma] + 1$, and $N + 1$ real numbers denoted $(t_i)_{0 \leqslant i \leqslant N}$, and defined as follows*:

$$t_0 = 0, \ \forall i \geqslant 1, \quad t_i = T - [N - i]\gamma,$$

(*this means that $t_N = T$, $t_{N-1} = T - \gamma$, $t_{N-2} = T - 2\gamma$,* etc. ...) *such that $\forall n \in \mathbf{N}^*$, $\forall 0 \leqslant i \leqslant N - 1$, $\forall t \in [t_i, t_{i+1}[$, and for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the problem*:

$$\begin{cases} \forall s \in [t, t_{i+1}], \\ X_s = \xi + \displaystyle\int_t^s f_n(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma_n(r, X_r, Y_r)\,\mathrm{d}B_r, \\ Y_s = \theta_n(t_{i+1}, X_{t_{i+1}}) + \displaystyle\int_s^{t_{i+1}} g_n(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_{i+1}} Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^{t_{i+1}} (|X_r|^2 + |Y_r|^2 + |Z_r|^2)\,\mathrm{d}r < \infty, \end{cases} \tag{2.3.1}$$

admits a unique solution denoted $(X_s^{n,t,i,\xi}, Y_s^{n,t,i,\xi}, Z_s^{n,t,i,\xi})_{s \in [t, t_{i+1}]}$. It satisfies almost surely:

$$\forall s \in [t, t_{i+1}], \quad Y_s^{n,t,i,\xi} = \theta_n(s, X_s^{n,t,i,\xi}), \tag{2.3.2}$$

$$\forall s \in [t, t_{i+1}], \quad Z_s^{n,t,i,\xi} = \nabla_x \theta_n(s, X_s^{n,t,i,\xi}) \sigma_n(s, X_s^{n,t,i,\xi}, Y_s^{n,t,i,\xi}), \tag{2.3.3}$$

$$\forall s \in [t, t_{i+1}], \quad |Z_s^{n,t,i,\xi}| \leqslant \Gamma', \tag{2.3.4}$$

where Γ' only depends on $k, \lambda, \Lambda, P, Q$ and T.

Proof. Let us fix an integer n and let us consider $i \in \{0, .., N - 1\}$, $t \in [t_i, t_{i+1}[$ and ξ a $\mathscr{G}_t$-measurable random vector with finite second moment. Thanks to Lemma 2.1, we can associate to every $n \in \mathbf{N}^*$ a process $(U_s^{n,t,\xi}, V_s^{n,t,\xi}, W_s^{n,t,\xi})_{t \leqslant s \leqslant T}$ satisfying both (2.1.6) (with $\tilde{\theta}$ replaced by θ_n and $\tilde{\sigma}$ by σ_n) and (2.3.1). Using Theorem 1.1 and the choice of γ, we show that this solution is unique. From (2.2.2), (2.2.3) and property (A2.2), we prove that (2.3.4) is satisfied and so complete the proof of Corollary 2.3. □

Proposition 2.4. *Under Assumption* (A2), *and keeping the notations of Proposition* 2.2, *there exists a map $\theta : [0, T] \times \mathbf{R}^P \to \mathbf{R}^Q$, such that*

(2.4.1) $\theta_n \to \theta$ *uniformly on every compact set of* $[0, T] \times \mathbf{R}^P$ *as* $n \to +\infty$,

(2.4.2) $\forall (t, x) \in [0, T] \times \mathbf{R}^P, \quad |\theta(t, x)| \leqslant C,$

(2.4.3) $\forall (t,x) \in [0,T] \times \mathbf{R}^P, \quad \forall (t',x') \in [0,T] \times \mathbf{R}^P,$

$$|\theta(t',x') - \theta(t,x)| \leqslant \Gamma |x'-x| + \kappa |t'-t|^{1/2},$$

(2.4.4) $\forall x \in \mathbf{R}^P, \quad \theta(T,x) = h(x).$

(2.4.5) $\forall i \in \{0,..,N-1\}$, $\forall t \in [t_i, t_{i+1}[$, *for every* $\mathscr{G}_t$*-measurable random vector* ξ *with finite second moment, the problem*

$$\begin{cases} \forall s \in [t, t_{i+1}], \\ X_s = \xi + \displaystyle\int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r, \\ Y_s = \theta(t_{i+1}, X_{t_{i+1}}) + \displaystyle\int_s^{t_{i+1}} g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_{i+1}} Z_r\,\mathrm{d}B_r, \\ \mathbf{E} \displaystyle\int_t^{t_{i+1}} (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s < \infty, \end{cases}$$

admits a unique solution denoted $(X_s^{t,i,\xi}, Y_s^{t,i,\xi}, Z_s^{t,i,\xi})_{t \leqslant s \leqslant t_{i+1}}$. *It satisfies*

$$\mathbf{P}\{\forall s \in [t, t_{i+1}], Y_s^{t,i,\xi} = \theta(s, X_s^{t,i,\xi})\} = 1,$$

and

$$\mathbf{P} \otimes \mu\{(\omega, s) \in \Omega \times [t, t_{i+1}], |Z_s^{t,i,\xi}(\omega)| > \Gamma'\} = 0,$$

where μ *stands for the Lebesgue measure on* $\mathbf{R}$.

Proof. We build the map θ by using a time-running-down induction. Indeed, thanks to Theorem 1.1, we show that $\forall t \in [t_{N-1}, T[$, for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall s \in [t, T], \\ X_s = \xi + \displaystyle\int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E} \displaystyle\int_t^T (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s < \infty, \end{cases}$$

admits a unique solution denoted $(X_s^{t,N-1,\xi}, Y_s^{t,N-1,\xi}, Z_s^{t,N-1,\xi})_{t \leqslant s \leqslant T}$. Following the first part, let us define the map

$$\theta : [t_{N-1}, T] \times \mathbf{R}^P \to \mathbf{R}^Q, \quad (t,x) \mapsto \theta(t,x) = Y_t^{t,N-1,x}.$$

So, from Corollary 1.5, we know that, almost surely:

$$\forall s \in [t,T], \quad Y_s^{t,N-1,\xi} = \theta(s, X_s^{t,N-1,\xi}).$$

Moreover, from Corollary 1.7 and Corollary 2.3, we know that the maps $(\theta_n)_{n\in\mathbf{N}}$ defined in Proposition 2.2 satisfy

$$\theta_n \to \theta \quad \text{as } n \to +\infty$$

uniformly on every compact set of $[t_{N-1}, T] \times \mathbf{R}^P$. In particular, from (2.2.3) and (2.2.4), we have that $\forall (t,x) \in [t_{N-1}, T] \times \mathbf{R}^P, \forall (t',x') \in [t_{N-1}, T] \times \mathbf{R}^P$,

$$|\theta(t',x') - \theta(t,x)| \leqslant \Gamma|x'-x| + \kappa|t'-t|^{1/2}.$$

Furthermore, Corollary 1.7 also proves that for every $t \in [t_{N-1}, T]$,

$$\mathbf{E}\int_t^T |Z_s^{n,t,N-1,\xi} - Z_s^{t,N-1,\xi}|^2 \,\mathrm{d}s \to 0 \quad \text{as } n \to +\infty.$$

Therefore, from Corollary 2.3, we deduce that

$$\mathbf{P}\otimes\mu\{(\omega,s)\in\Omega\times[t,T], |Z_s^{t,N-1,\xi}(\omega)| > \Gamma'\} = 0.$$

We have just proved that (2.4.1)–(2.4.5) were satisfied on $[t_{N-1}, T]$.

Let us show that the same can be done on the interval $[t_{N-2}, t_{N-1}]$.

We know that the functions f, g, σ and $\theta(t_{N-1}, .)$ satisfy Assumption (A1) with $\tilde{K}$ as Lipschitz-monotonicity constant. So, applying Theorem 1.1, we show that for every $t \in [t_{N-2}, t_{N-1}]$, for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall s \in [t, t_{N-1}], \\ X_s = \xi + \displaystyle\int_t^s f(r, X_r, Y_r, Z_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r)\,\mathrm{d}B_r, \\ Y_s = \theta(t_{N-1}, X_{t_{N-1}}) + \displaystyle\int_s^{t_{N-1}} g(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_s^{t_{N-1}} Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^{t_{N-1}} (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\,\mathrm{d}s < \infty, \end{cases}$$

admits a unique solution denoted $(X_s^{t,N-2,\xi}, Y_s^{t,N-2,\xi}, Z_s^{t,N-2,\xi})_{t_{N-2}\leqslant s\leqslant t_{N-1}}$. From Corollary 1.5, we can define

$$\theta : [t_{N-2}, t_{N-1}] \times \mathbf{R}^P \to \mathbf{R}^Q, \quad (t,x) \mapsto \theta(t,x) = Y_t^{t,N-2,x}.$$

Note that both definitions of $\theta(t_{N-1}, .)$ are compatible.

So, from Corollary 1.5, we know that, almost surely:

$$\forall s \in [t, t_{N-1}], \quad Y_s^{t,N-2,\xi} = \theta(s, X_s^{t,N-2,\xi}).$$

Moreover, we know that $(\theta_n(t_{N-1}, .))_{n\in\mathbf{N}}$ converges uniformly on every compact set to $\theta(t_{N-1}, .)$. So, from Corollary 1.7, property (1.7.2),

$$\theta_n \to \theta \quad \text{as } n \to +\infty,$$

uniformly on every compact set of $[t_{N-2}, t_{N-1}] \times \mathbf{R}^P$, and therefore on every compact set of $[t_{N-2}, T] \times \mathbf{R}^P$. In particular, $\forall (t,x) \in [t_{N-2}, T] \times \mathbf{R}^P$, $\forall (t',x') \in [t_{N-2}, T] \times \mathbf{R}^P$,

$$|\theta(t',x') - \theta(t,x)| \leqslant \Gamma|x'-x| + \kappa|t'-t|^{1/2}.$$

Furthermore, Corollary 1.7, (1.7.1), also proves that for every $t \in [t_{N-2}, t_{N-1}[$,

$$\mathbf{E} \int_t^{t_{N-1}} |Z_s^{n,t,N-2,\xi} - Z_s^{t,N-2,\xi}|^2 \, \mathrm{d}s \to 0 \quad \text{as } n \to +\infty.$$

So, from Corollary 2.3, property (2.3.4), this shows that

$$\mathbf{P} \otimes \mu\{(\omega, s) \in \Omega \times [t, t_{N-1}], |Z_s^{t,N-2,\xi}(\omega)| > \Gamma'\} = 0.$$

We have just proved that (2.4.1)–(2.4.5) were satisfied on $[t_{N-2}, t_{N-1}]$. Therefore, using a running-down induction, we build a map θ satisfying Proposition 2.4. □

Corollary 2.5. *Under Assumption* (A2), *and keeping the notations of Proposition* 2.4, *for every* $t \in [0, T[$, *for every* $\mathscr{G}_t$*-measurable random vector* ξ *with finite second moment*, *every solution* $(X_s, Y_s, Z_s)_{t \leqslant s \leqslant T}$ *of the problem*

$$\begin{cases} \forall s \in [t, T], \\ X_s = \xi + \displaystyle\int_t^s f(r, X_r, Y_r, Z_r) \, \mathrm{d}r + \int_t^s \sigma(r, X_r, Y_r) \, \mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g(r, X_r, Y_r, Z_r) \, \mathrm{d}r - \int_s^T Z_r \, \mathrm{d}B_r, \\ \mathbf{E} \displaystyle\int_t^T (|X_s|^2 + |Y_s|^2 + |Z_s|^2) \, \mathrm{d}s < \infty, \end{cases} \tag{2.5.1}$$

satisfies for every $i \leqslant j \leqslant N-1$,

$$\begin{aligned} \mathbf{E} \sup_{\tilde{t}_j \leqslant s \leqslant \tilde{t}_{j+1}} |X_s - X_s^{\tilde{t}_j, j, X_{\tilde{t}_j}}|^2 &= \mathbf{E} \sup_{\tilde{t}_j \leqslant s \leqslant \tilde{t}_{j+1}} |Y_s - Y_s^{\tilde{t}_j, j, X_{\tilde{t}_j}}|^2 \\ &= \mathbf{E} \int_{\tilde{t}_j}^{\tilde{t}_{j+1}} |Z_s - Z_s^{\tilde{t}_j, j, X_{\tilde{t}_j}}|^2 \, \mathrm{d}s = 0, \end{aligned} \tag{2.5.2}$$

where i is the unique integer of $\{0, .., N-1\}$ *such that* $t \in [t_i, t_{i+1}[$, *and* $(\tilde{t}_j)_{i \leqslant j \leqslant N}$ *stand for the real numbers defined as follows*:

$$\tilde{t}_i = t, \qquad \tilde{t}_j = t_j \quad \text{if } j > i.$$

In particular,

$$\mathbf{P}\{\forall s \in [t, T], \; Y_s = \theta(s, X_s)\} = 1, \tag{2.5.3i}$$

and

$$\mathbf{P} \otimes \mu\{(\omega, s) \in \Omega \times [t, T], \; |Z_s(\omega)| > \Gamma'\} = 0. \tag{2.5.3ii}$$

Proof. We apply the same running-down induction method as in Proposition 2.4. Indeed, let us consider $t \in [0, T[$, and let us denote i the integer of $\{0, .., N-1\}$ such that $t \in [t_i, t_{i+1}[$.

If $i=N-1$, then (2.5.2), (2.5.3i) and (2.5.3ii) are direct consequences of Proposition 2.4.

If $i \leqslant N-2$, and if $(X_s,Y_s,Z_s)_{t \leqslant s \leqslant T}$ is a solution of the problem (2.5.1), then $(X_s,Y_s,Z_s)_{t_{N-1} \leqslant s \leqslant T}$ is a solution of the problem

$$\begin{cases} \forall s \in [t_{N-1},T], \\ X_s = X_{t_{N-1}} + \displaystyle\int_{t_{N-1}}^{s} f(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_{t_{N-1}}^{s} \sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_{t_{N-1}}^{T} (|X_s|^2+|Y_s|^2+|Z_s|^2)\,\mathrm{d}s < \infty. \end{cases}$$

So, from Proposition 2.4,

$$\begin{aligned} \mathbf{E} \sup_{t_{N-1}\leqslant s\leqslant T} |X_s - X_s^{t_{N-1},N-1,X_{t_{N-1}}}|^2 &= \mathbf{E} \sup_{t_{N-1}\leqslant s\leqslant T} |Y_s - Y_s^{t_{N-1},N-1,X_{t_{N-1}}}|^2 \\ &= \mathbf{E}\int_{t_{N-1}}^{T} |Z_s - Z_s^{t_{N-1},N-1,X_{t_{N-1}}}|^2\,\mathrm{d}s = 0. \end{aligned}$$

In particular,

$$Y_{t_{N-1}} = \theta(t_{N-1},X_{t_{N-1}}) \quad \text{a.s.}$$

Therefore, $(X_s,Y_s,Z_s)_{t\leqslant s\leqslant t_{N-1}}$ is the solution of the problem

$$\begin{cases} \forall s \in [t,t_{N-1}], \\ X_s = \xi + \displaystyle\int_t^s f(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_t^s \sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = \theta(t_{N-1},X_{t_{N-1}}) + \displaystyle\int_s^{t_{N-1}} g(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^{t_{N-1}} Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^{t_{N-1}} (|X_s|^2+|Y_s|^2+|Z_s|^2)\,\mathrm{d}s < \infty. \end{cases}$$

Hence, an induction proves (2.5.2). Using (2.4.5), we easily deduce (2.5.3i) and (2.5.3ii). □

Theorem 2.6 (Existence and uniqueness of solutions). *We assume that Assumption* (A2) *is satisfied, and we keep the notations of Proposition* 2.4. *Then,*

(2.6.1) $\forall T>0$, *for every* $\mathscr{G}_0$*-measurable random vector* ξ *with finite second moment, the problem* (**E**) *admits a unique solution.*

(2.6.2) $\forall t \in [0,T]$, *for every* $\mathscr{G}_t$*-measurable random vector* ξ *with finite second moment, the unique solution of the problem*

$$\begin{cases} \forall s \in [t,T], \\ X_s^{t,\xi} = \xi + \int_t^s f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\xi}, Y_r^{t,\xi})\,\mathrm{d}B_r, \\ Y_s^{t,\xi} = h(X_T^{t,\xi}) + \int_s^T g(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})\,\mathrm{d}r - \int_s^T Z_r^{t,\xi}\,\mathrm{d}B_r, \\ \mathbf{E}\int_t^T (|X_s^{t,\xi}|^2 + |Y_s^{t,\xi}|^2 + |Z_s^{t,\xi}|^2)\,\mathrm{d}s < \infty, \end{cases} \tag{2.6.2}$$

satisfies

$$\mathbf{P}\{\forall s \in [t,T],\ Y_s^{t,\xi} = \theta(s, X_s^{t,\xi})\} = 1, \tag{2.6.3i}$$

and

$$\mathbf{P} \otimes \mu\{(\omega, s) \in \Omega \times [t,T],\ |Z_s^{t,\xi}(\omega)| > \Gamma'\} = 0. \tag{2.6.3ii}$$

In particular, there exist versions of the processes $(Y_s^{t,\xi})_{t \leqslant s \leqslant T}$ *and* $(Z_s^{t,\xi})_{t \leqslant s \leqslant T}$ *whose trajectories are uniformly bounded.*

Remark 2.7. The same reasoning as the one done in Remark 1.6 shows that the map θ only depends on f, g, h, σ and T.

Proof of Theorem 2.6. Let us consider a $\mathscr{G}_0$-measurable random vector ξ with finite second moment. Let us show existence of a solution to the problem (E). We now use a running-up induction. Indeed, thanks to Proposition 2.4, the problem

$$\begin{cases} \forall 0 \leqslant t \leqslant t_1, \\ X_t = \xi + \int_0^t f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\,\mathrm{d}B_s, \\ Y_t = \theta(t_1, X_{t_1}) + \int_t^{t_1} g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^{t_1} Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\int_0^{t_1} (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty, \end{cases}$$

admits a unique solution, which was denoted in (2.4.5) $(X^{0,0,\xi}, Y^{0,0,\xi}, Z^{0,0,\xi})$, and which is now denoted $(X_t^{(0)}, Y_t^{(0)}, Z_t^{(0)})_{0 \leqslant t \leqslant t_1}$.

Then, using once again Proposition 2.4, the problem

$$\begin{cases} \forall t_1 \leqslant t \leqslant t_2, \\ X_t = X_{t_1}^{(0)} + \int_{t_1}^{t} f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_{t_1}^{t} \sigma(s, X_s, Y_s)\,\mathrm{d}B_s, \\ Y_t = \theta(t_2, X_{t_2}) + \int_{t}^{t_2} g(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_{t}^{t_2} Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\int_{t_1}^{t_2} (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty, \end{cases}$$

admits a unique solution, which was denoted in (2.4.5) $(X^{t_1,1,X_{t_1}^{(0)}}, Y^{t_1,1,X_{t_1}^{(0)}}, Z^{t_1,1,X_{t_1}^{(0)}})$, and which is now denoted $(X_t^{(1)}, Y_t^{(1)}, Z_t^{(1)})_{t_1 \leqslant t \leqslant t_2}$. It satisfies

$$X_{t_1}^{(1)} = X_{t_1}^{(0)} \quad \text{a.s.}$$

$$Y_{t_1}^{(1)} = \theta(t_1, X_{t_1}^{(0)}) = Y_{t_1}^{(0)} \quad \text{a.s.}$$

We then build, using an induction, the processes $((X_t^{(k)}, Y_t^{(k)}, Z_t^{(k)})_{t_k \leqslant t \leqslant t_{k+1}})_{k \in \{1,..,N-1\}}$, solutions of the problems:

$$\begin{cases} \forall t_k \leqslant t \leqslant t_{k+1}, \\ X_t = X_{t_k}^{(k-1)} + \int_{t_k}^{t} f(s, X_s^{(k)}, Y_s^{(k)}, Z_s^{(k)})\,\mathrm{d}s + \int_{t_k}^{t} \sigma(s, X_s^{(k)}, Y_s^{(k)})\,\mathrm{d}B_s, \\ Y_t^{(k)} = \theta(t_{k+1}, X_{t_{k+1}}^{(k)}) + \int_{t}^{t_{k+1}} g(s, X_s^{(k)}, Y_s^{(k)}, Z_s^{(k)})\,\mathrm{d}s - \int_{t}^{t_{k+1}} Z_s^{(k)}\,\mathrm{d}B_s, \\ \mathbf{E}\int_{t_k}^{t_{k+1}} (|X_t^{(k)}|^2 + |Y_t^{(k)}|^2 + |Z_t^{(k)}|^2)\,\mathrm{d}t < \infty. \end{cases}$$

We have for every $1 \leqslant k \leqslant N-1$,

$$X_{t_k}^{(k)} = X_{t_k}^{(k-1)} \quad \text{a.s.}; \quad Y_{t_k}^{(k)} = \theta(t_k, X_{t_k}^{(k-1)}) = Y_{t_k}^{(k-1)} \quad \text{a.s.}$$

This proves that the process $(X_t, Y_t, Z_t)_{0 \leqslant t \leqslant T}$ defined as follows:

$$\forall k \in \{0,..,N-1\},\ \forall t \in [t_k, t_{k+1}], \quad X_t = X_t^{(k)}, \quad Y_t = Y_t^{(k)}, \quad Z_t = Z_t^{(k)},$$

is a solution of the problem. This shows existence of a solution to (E).

Let us prove uniqueness of this solution. Let us consider a solution $(U_t, V_t, W_t)_{0 \leqslant t \leqslant T}$ of the problem (E). Then, Corollary 2.5 shows that

$$\mathbf{E}\sup_{0 \leqslant t \leqslant t_1} |U_t - X_t|^2 = \mathbf{E}\sup_{0 \leqslant t \leqslant t_1} |U_t - X_t|^2 = \mathbf{E}\int_0^{t_1} |W_s - Z_s|^2\,\mathrm{d}s = 0.$$

In particular,

$$U_{t_1} = X_{t_1} \quad \text{a.s.}$$

A new induction completes the proof of (2.6.1).

Finally, (2.6.3i) and (2.6.3ii) are direct consequences of Corollary 2.5. This completes the proof of Theorem 2.6. □

Corollary 2.8 (Regularity upon the initial conditions). *Assume that Assumption* (A2) *is in force. Then, there exists a constant C^* only depending on $k, K, \lambda, \Lambda, P, Q$ and T, such that $\forall (t,x) \in [0,T] \times \mathbf{R}^P$, $\forall (t',x') \in [0,T] \times \mathbf{R}^P$,*

$$\mathbf{E}\left(\sup_{0 \leqslant s \leqslant T} |X_s^{t,x} - X_s^{t',x'}|^2 + \sup_{0 \leqslant s \leqslant T} |Y_s^{t,x} - Y_s^{t',x'}|^2 + \int_0^T |Z_s^{t,x} - Z_s^{t',x'}|^2 \, \mathrm{d}s\right)$$
$$\leqslant C^*(|x - x'|^2 + |t - t'|).$$

Proof. Let us consider $(t,x) \in [0,T] \times \mathbf{R}^P$ and $(t',x') \in [0,T] \times \mathbf{R}^P$. Using property (A2.2) as well as (2.4.2), (2.4.3), (2.6.3i) and (2.6.3ii), we prove that there exists a constant C^* only depending on k, K, λ, Λ, P, Q and T such that $\forall (t,x) \in [0,T] \times \mathbf{R}^P, \forall (t',x') \in [0,T] \times \mathbf{R}^P$,

$$\mathbf{E}\left(\sup_{0 \leqslant s \leqslant t \vee t'} |X_s^{t,x} - X_s^{t',x'}|^2 + \sup_{0 \leqslant s \leqslant t \vee t'} |Y_s^{t,x} - Y_s^{t',x'}|^2 + \int_0^{t \vee t'} |Z_s^{t,x} - Z_s^{t',x'}|^2 \, \mathrm{d}s\right)$$
$$\leqslant C^*(|x - x'|^2 + |t - t'|).$$

Moreover, applying successively Theorem 1.3 on each small interval $[\tilde{t}_j, \tilde{t}_{j+1}]$, $i \leqslant j \leqslant N$, as done in Theorem 2.6, where i stands for the integer of $\{0, .., N-1\}$ such that $t \vee t' \in [t_i, t_{i+1}[$ and $(\tilde{t}_j)_{i \leqslant j \leqslant N}$ stand for the real numbers defined by $\tilde{t}_i = t$, $\tilde{t}_j = t_j$ if $j > i$, and modifying C^* if necessary, we prove that,

$$\mathbf{E}\left(\sup_{t \vee t' \leqslant s \leqslant T} |X_s^{t,x} - X_s^{t',x'}|^2 + \sup_{t \vee t' \leqslant s \leqslant T} |Y_s^{t,x} - Y_s^{t',x'}|^2 + \int_{t \vee t'}^T |Z_s^{t,x} - Z_s^{t',x'}|^2 \, \mathrm{d}s\right)$$
$$\leqslant C^*(|x - x'|^2 + |t - t'|).$$

This completes the proof. □

3. A locally lipschitzian-monotonous coefficients case

Thanks to properties (2.6.3i) and (2.6.3ii), we are able in this section to relax several assumptions required in Theorem 2.6 on the coefficients with respect to the variables y and z. Indeed, in Section 3.1, we improve Theorem 2.6 to a class of systems with locally lipschitzian-monotonous coefficients, whereas Section 3.2 extends this latter result to a set of non-standard FBSDEs.

3.1. *The standard case*

In this section, we are typically interested in systems of the following form:

$$\forall t \in [0,T],$$
$$X_t = \xi + \int_0^t f_1(X_s, Y_s, Z_s) \, \mathrm{d}s + \int_0^t f_2(s, X_s, Y_s, Z_s) Z_s \, \mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s) \, \mathrm{d}B_s,$$

$$Y_t = h(X_T) + \int_t^T g_1(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g_2(s, X_s, Y_s, Z_s) Z_s\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s,$$

$$\mathbf{E}\int_0^T (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty.$$

Actually, we prove here a result of existence and uniqueness for systems of type (E) whose coefficients satisfy the following assumption:

Assumption (A3). We say that the functions f, g, h and σ satisfy Assumption (A3) if there exist three non-decreasing functions $k, K, \Lambda : \mathbf{R}^+ \to \mathbf{R}^+$, and a non-increasing function $\lambda : \mathbf{R}^+ \to \mathbf{R}^+ \setminus \{0\}$, such that they satisfy both (A1.0) and the following properties:

(A3.1): $\forall t \in [0,T], \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and $\forall (x', y', z') \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant K(|y| + |y'| + |z|)(1 + |z'|) \quad (|y - y'| + |z - z'|),$$

$$|g(t,x,y,z) - g(t,x',y,z')| \leqslant K(|y| + |z|)(1 + |z'|) \quad (|x - x'| + |z - z'|),$$

$$|h(x) - h(x')| \leqslant k(0) \quad |x - x'|,$$

$$|\sigma(t,x,y) - \sigma(t,x',y')|^2 \leqslant k^2(|y| + |y'|) \quad (|x - x'|^2 + |y - y'|^2).$$

(A3.2): $\forall t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and $\forall (x', y') \in \mathbf{R}^P \times \mathbf{R}^Q$,

$$\langle x - x', f(t,x,y,z) - f(t,x',y,z)\rangle \leqslant K(|y| + |z|)|x - x'|^2,$$

$$\langle y - y', g(t,x,y,z) - g(t,x,y',z)\rangle \leqslant K(|y| + |y'| + |z|)|y - y'|^2.$$

(A3.3): $\forall t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z)| \leqslant \Lambda(|y|)(1 + |z|),$$

$$|g(t,x,y,z)| \leqslant \Lambda(0)(1 + |y| + |z|),$$

$$|\sigma(t,x,y)| \leqslant \Lambda(|y|),$$

$$|h(x)| \leqslant \Lambda(0).$$

(A3.4): $\forall t \in [0,T],\ \forall (x,y) \in \mathbf{R}^P$,

$$\forall \xi \in \mathbf{R}^P, \quad \langle \xi, a(t,x,y)\xi\rangle \geqslant \lambda(|y|)|\xi|^2.$$

(A3.5): $\forall t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, the functions $u \mapsto f(t,u,y,z)$ and $v \mapsto g(t,x,v,z)$ are continuous. Moreover, σ is continuous on its definition set.

Remark. From (A3.1) and (A3.3), we deduce that there exists a non-increasing function $\bar{K} : \mathbf{R}^+ \to \mathbf{R}^+$, such that

(A3.1′): $\forall t \in [0,T],\ \forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and $\forall (x, y', z') \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant \bar{K}(|y| + |y'| + |z|) \quad (|y - y'| + |z - z'|),$$

$$|g(t,x,y,z) - g(t,x',y,z')| \leqslant \bar{K}(|y| + |z|) \quad (|x - x'| + |z - z'|).$$

Indeed, $\forall (t,x,y,z)\in[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$ and $\forall (y',z')\in\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$,

$$\begin{aligned}&\left|\frac{1}{1+|z|}\,f(t,x,y,z)-\frac{1}{1+|z'|}\,f(t,x,y',z')\right|\\&\quad\leqslant\frac{|z-z'|}{(1+|z|)(1+|z'|)}\,|f(t,x,y,z)|+\frac{1}{1+|z'|}\,|f(t,x,y,z)-f(t,x,y',z')|\\&\quad\leqslant\Lambda(|y|)|z-z'|+K(|y|+|y'|+|z|)(|y-y'|+|z-z'|).\end{aligned}$$

Therefore,

$$\begin{aligned}&|f(t,x,y,z)-f(t,x,y',z')|\\&\quad=\left|(1+|z|)\,\frac{1}{1+|z|}\,f(t,x,y,z)-(1+|z'|)\,\frac{1}{1+|z'|}\,f(t,x,y',z')\right|\\&\quad\leqslant(1+|z|)(\Lambda(|y|)+K(|y|+|y'|+|z|))(|y-y'|+|z-z'|)+\Lambda(|y|)|z-z'|.\end{aligned}$$

Of the course, the same scheme holds for g.

Theorem 3.1 (Existence of solutions). *Under* (A3), *there exist two lipschitzian maps* $v^{(1)}:\mathbf{R}^Q\to\mathbf{R}^Q$ *and* $v^{(2)}:\mathbf{R}^{Q\times P}\to\mathbf{R}^{Q\times P}$, *null outside a compact set*, *satisfying*

$$\forall y\in\mathbf{R}^Q,\quad |v^{(1)}(y)|\leqslant|y|,\qquad \forall z\in\mathbf{R}^{Q\times P},\quad |v^{(2)}(z)|\leqslant|z|,$$

such that for every $t\in[0,T]$, *for every* $\mathscr{G}_t$*-measurable random vector* ξ *with finite second moment*, *the unique solution denoted* $(U_s^{t,\xi},V_s^{t,\xi},W_s^{t,\xi})_{t\leqslant s\leqslant T}$ *of the problem*

$$\begin{cases}\forall s\in[t,T],\\ X_s=\xi+\displaystyle\int_t^s f(u,X_u,v^{(1)}(Y_u),v^{(2)}(Z_u))\,\mathrm{d}u+\int_t^s\sigma(u,X_u,v^{(1)}(Y_u))\,\mathrm{d}B_u,\\ Y_s=h(X_T)+\displaystyle\int_s^T g(u,X_u,v^{(1)}(Y_u),v^{(2)}(Z_u))\,\mathrm{d}u-\int_s^T Z_u\,\mathrm{d}B_u,\\ \mathbf{E}\displaystyle\int_t^T(|X_u|^2+|Y_u|^2+|Z_u|^2)\,\mathrm{d}u<\infty\end{cases}$$

is also a solution of the problem (2.6.2).

Hence $(U_s^{t,\xi},V_s^{t,\xi},W_s^{t,\xi})_{t\leqslant s\leqslant T}$ *satisfy the conclusions of Theorem* 2.6. *In particular*, *there exist a map* $\theta:[0,T]\times\mathbf{R}^P\to\mathbf{R}^Q$, *as well as a constant* C_0 *only depending on* $\Lambda(0)$ *and* T *and a constant* C_1, *only depending on* $k(2C_0),\lambda(C_0),\Lambda(C_0),P,Q$ *and* T, *such that*

$$\begin{aligned}&\forall (t,x)\in[0,T]\times\mathbf{R}^P,\quad |\theta(t,x)|\leqslant C_0,\\&\forall t\in[0,T],\ \forall (x,x')\in(\mathbf{R}^P)^2,\quad |\theta(t,x')-\theta(t,x)|\leqslant C_1|x-x'|.\end{aligned}\tag{3.1.2}$$

Moreover, *for every* $t\in[0,T]$, *for every* $\mathscr{G}_t$*-measurable random vector* ξ *with finite second moment*, $(U_s^{t,\xi},V_s^{t,\xi},W_s^{t,\xi})_{t\leqslant s\leqslant T}$ *satisfies*

$$\mathbf{P}\{\forall s\in[t,T],V_s^{t,\xi}=\theta(s,U_s^{t,\xi})\}=1,\tag{3.1.3i}$$

and

$$\mathbf{P} \otimes \mu\{(\omega,s) \in \Omega \times [t,T], |W_s^{t,\xi}(\omega)| > C_2\} = 0, \tag{3.1.3ii}$$

where C_2 only depends on $k(2C_0), \lambda(C_0), \Lambda(C_0), P, Q$ and T.

Proof. For every $L \in \mathbf{N}^*$, we define $(v_{L,n})_{n\in\mathbf{N}^*}$, sequence of functions from $\mathbf{R}^L$ into $\mathbf{R}^L$, by

$$v_{L,n}(u) = \begin{cases} u & \text{if } |u| \leqslant n, \\ 0 & \text{if } |u| \geqslant 2n, \\ \frac{2n-|u|}{n} u & \text{if } n \leqslant |u| \leqslant 2n, \end{cases}$$

whose following properties are easily verified

$$\forall n \in \mathbf{N}^*, \ \ \forall u \in \mathbf{R}^L, \quad |v_{L,n}(u)| \leqslant |u|,$$

$$\forall n \in \mathbf{N}^*, \ \ \forall u \in \mathbf{R}^L, \quad |v_{L,n}(u)| \leqslant 2n,$$

$$\forall n \in \mathbf{N}^*, \ \ \forall (u,u') \in (\mathbf{R}^L)^2, \quad |v_{L,n}(u) - v_{L,n}(u')| \leqslant 3|u-u'|.$$

Moreover, we let for every $n \in \mathbf{N}^*$,

$$v_n^{(1)} = v_{Q,n}, \quad v_n^{(2)} = v_{Q\times P,n},$$

and $\forall (t,x,y,z) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$f^{(n)}(t,x,y,z) = f(t,x,v_n^{(1)}(y), v_n^{(2)}(z)),$$

$$g^{(n)}(t,x,y,z) = g(t,x,v_n^{(1)}(y), v_n^{(2)}(z)),$$

$$\sigma^{(n)}(t,x,y) = \sigma(t,x,v_n^{(1)}(y)).$$

Hence, for every $n \in \mathbf{N}^*$, there exist four constants K_n, k_n, Λ_n and $\lambda_n > 0$, such that $f^{(n)}, g^{(n)}, h$ and $\sigma^{(n)}$ satisfy Assumption (A2) with respect to the constants K_n, k_n, Λ_n and λ_n.

Therefore, $\forall n \in \mathbf{N}^*, \forall t \in [0,T]$, and for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall s \in [t,T], \\ X_s = \xi + \displaystyle\int_t^s f^{(n)}(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_t^s \sigma^{(n)}(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g^{(n)}(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^T (|X_r|^2 + |Y_r|^2 + |Z_r|^2)\,\mathrm{d}r < \infty, \end{cases}$$

admits from Theorem 2.6 a unique solution denoted $(X_s^{n,t,\xi}, Y_s^{n,t,\xi}, Z_s^{n,t,\xi})_{t\leqslant s\leqslant T}$.

Moreover, following the proof of (2.1.7), there exists a constant C_0 only depending on $\Lambda(0)$ and T, such that for every $n \in \mathbf{N}^*$, for every $(t,x) \in [0,T] \times \mathbf{R}^P$, $|Y_t^{n,t,x}| \leqslant C_0$. Therefore, from (2.6.3i):

$$\forall n \in \mathbf{N}^*, \quad \mathbf{P}\left\{\sup_{t \leqslant s \leqslant T} |Y_s^{n,t,\xi}(\omega)| \leqslant C_0\right\} = 1.$$

Therefore, from Theorem 2.6, there exists a constant C_2, only depending on $k(2C_0)$, $\lambda(C_0), \Lambda(C_0), P, Q$ and T, such that

$$\forall n \in \mathbf{N}^*, \quad \mathbf{P} \otimes \mu\{(\omega,s) \in \Omega \times [t,T], |Z_s^{n,t,\xi}(\omega)| > C_2\} = 0.$$

Hence, for $n > \max(C_0, C_2)$, we have for every $t \in [0,T]$ and for every $\mathscr{G}_t$-measurable random vector ξ with finite second moment

$$\mathbf{P}\{\forall s \in [t,T], v_n^{(1)}(Y_s^{n,t,\xi}) = Y_s^{n,t,\xi}\} = 1,$$

$$\mathbf{P} \otimes \mu\{(\omega,s) \in \Omega \times [t,T], v^{(2)}(Z_s^{n,t,\xi}(\omega)) \neq Z_s^{n,t,\xi}(\omega)\} = 0.$$

Finally, choosing $n > \max(C_0, C_2), (X_s^{n,t,\xi}, Y_s^{n,t,\xi}, Z_s^{n,t,\xi})_{t \leqslant s \leqslant T}$ is a solution of the problem (2.6.2). This completes the proof of Theorem 3.1. □

Theorem 3.2 (Uniqueness of the solution). *Under Assumption* (A3), *problem* (E) *admits a unique solution* (*given by Theorem* 3.1).

Remark. We firstly note, as we have done for Theorem 1.1, that under (A3), every $\{\mathscr{G}_t\}_{0 \leqslant t \leqslant T}$-progressively measurable solution $(X_t, Y_t, Z_t)_{0 \leqslant t \leqslant T}$ of the problem (**E**) satisfies

(i) $(Y_t)_{0 \leqslant t \leqslant T}$ is continuous and satisfies $\mathbf{E} \sup_{0 \leqslant t \leqslant T} |Y_t|^2 < \infty$.
(ii) Following the proof of (2.1.7) (take for every $s \in [0,T]$ the conditional expectation given $\mathscr{G}_s$ instead of the conditional expectation given $\mathscr{G}_t$), and using the continuity of Y, we deduce $\mathbf{P}\{\sup_{t \leqslant s \leqslant T} |Y_s| \leqslant C_0\} = 1$.
(iii) From Assumption (A3), $(X_t)_{0 \leqslant t \leqslant T}$ is continuous and satisfies $\mathbf{E} \sup_{0 \leqslant t \leqslant T} |X_t|^2 < \infty$.

Let us prove Theorem 3.2.

Proof. We use the same kind of estimates on small intervals as the ones used in the first part as well as a running-down induction method as used in the second part. Furthermore, we keep the notations introduced in Theorem 3.1.

Let ξ be a $\mathscr{G}_0$ measurable random variable with finite second moment and $(X_s, Y_s, Z_s)_{0 \leqslant s \leqslant T}$ a solution of the problem (**E**). The proof is then based on the following lemma:

Lemma 3.3. *There exists a real* $\delta > 0$, *only depending on* k, K, Λ, C_0, C_1 *and* C_2 *such that*

$$\forall t \in [T-\delta, T], \mathbf{E} \sup_{t \leqslant s \leqslant T} |X_s - U_s^t|^2 = \mathbf{E} \sup_{t \leqslant s \leqslant T} |Y_s - V_s^t|^2 = \mathbf{E} \int_t^T |Z_s - W_s^t|^2 \, ds = 0,$$

where for every $t\in[0,T],(U_s^t,V_s^t,W_s^t)_{t\leqslant s\leqslant T}$ *is the solution of*

$$\begin{cases}\forall s\in[t,T],\\ U_s^t=X_t+\displaystyle\int_t^s f(r,U_r^t,v^{(1)}(V_r^t),v^{(2)}(W_r^t))\,\mathrm{d}r+\int_t^s \sigma(r,U_r^t,v^{(1)}(V_r^t))\,\mathrm{d}B_r,\\ V_s^t=h(U_T^t)+\displaystyle\int_s^T g(r,U_r^t,v^{(1)}(V_r^t),v^{(2)}(W_r^t))\,\mathrm{d}r-\int_s^T W_r^t\,\mathrm{d}B_r,\\ \mathbf{E}\displaystyle\int_t^T(|U_r^t|^2+|V_r^t|^2+|W_r^t|^2)\,\mathrm{d}r<\infty.\end{cases}$$

Proof. Let us consider $t\in[0,T]$. Then, from Theorem 3.1, $(U_s^t,V_s^t,Z_s^t)_{t\leqslant s\leqslant T}$ satisfies

$$\begin{cases}\forall s\in[t,T],\\ U_s^t=X_t+\displaystyle\int_t^s f(r,U_r^t,V_r^t,W_r^t)\,\mathrm{d}r+\int_t^s \sigma(r,U_r^t,V_r^t)\,\mathrm{d}B_r,\\ V_s^t=h(U_T^t)+\displaystyle\int_s^T g(r,U_r^t,V_r^t,W_r^t)\,\mathrm{d}r-\int_s^T W_r^t\,\mathrm{d}B_r,\\ \mathbf{E}\displaystyle\int_t^T(|U_r^t|^2+|V_r^t|^2+|W_r^t|^2)\,\mathrm{d}r<\infty.\end{cases}$$

Hence, from Itô's calculus, $\forall s\in[t,T]$,

$$\begin{aligned}\mathbf{E}|U_s^t-X_s|^2=2\mathbf{E}\int_t^s\langle U_r^t-X_r,f(r,U_r^t,V_r^t,W_r^t)-f(r,X_r,Y_r,Z_r)\rangle\,\mathrm{d}r\\ +\mathbf{E}\int_t^s|\sigma(r,U_r^t,V_r^t)-\sigma(r,X_r,Y_r)|^2\,\mathrm{d}r.\end{aligned}$$

Therefore, thanks to Assumption (A3) (in particular (A3.1′)), to (3.1.2) and (3.1.3), and to remark (ii) of the statement, there exists a constant c, only depending on k,K,Λ,C_0,C_1 and C_2 such that $\forall s\in[t,T]$,

$$\begin{aligned}\mathbf{E}|U_s^t-X_s|^2\leqslant c\Bigg[\mathbf{E}\int_t^s|U_r^t-X_r|(|U_r^t-X_r|+|V_r^t-Y_r|+|W_r^t-Z_r|)\,\mathrm{d}r\\ +\mathbf{E}\int_t^s(|U_r^t-X_r|^2+|V_r^t-Y_r|^2)\,\mathrm{d}r\Bigg].\end{aligned}$$

So, using standard estimates (see Section 1), we can find α only depending on k,K,Λ, C_0,C_1 and C_2, such that

$$\mathbf{E}|U_s^t-X_s|^2\leqslant\alpha\mathbf{E}\int_t^s(|U_r^t-X_r|^2+|V_r^t-Y_r|^2)\,\mathrm{d}r+\frac{1}{4}\frac{1}{C_1^2}\mathbf{E}\int_t^s|W_r^t-Z_r|^2\,\mathrm{d}r.\tag{3.3.1}$$

Moreover, Itô's formula also shows that $\forall s \in [t,T]$,

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}\int_s^T |Z_r - W_r^t|^2\,\mathrm{d}r$$
$$= \mathbf{E}|h(X_T) - h(U_T^t)|^2 + 2\mathbf{E}\int_s^T \langle Y_r - V_r^t, g(r,X_r,Y_r,Z_r) - g(r,U_r^t,V_r^t,W_r^t)\rangle\,\mathrm{d}r.$$

Hence, thanks to Assumptions (A3), to (3.1.2) and (3.1.3) and to remark (ii) of Theorem 3.2, we have, modifying c if necessary, $\forall s \in [t,T]$:

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}\int_s^T |Z_r - W_r^t|^2\,\mathrm{d}r$$
$$\leqslant C_1^2\mathbf{E}|X_T - U_T^t|^2 + c\left[\mathbf{E}\int_s^T |Y_r - V_r^t|(|X_r - U_r^t| + |Y_r - V_r^t| + |Z_r - W_r^t|)\,\mathrm{d}r\right].$$

From (3.3.1), there exists a constant α' only depending on k, K, Λ, C_0, C_1 and C_2, such that $\forall s \in [t,T]$,

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}\int_t^T |Z_r - W_r^t|^2\,\mathrm{d}r \leqslant \alpha'\mathbf{E}\int_t^T (|U_r^t - X_r|^2 + |V_r^t - Y_r|^2)\,\mathrm{d}r. \tag{3.3.2}$$

Finally, injecting (3.3.2) in (3.3.1), there exits a constant α'' only depending on k, K, Λ, C_0, C_1 and C_2, such that $\forall s \in [t,T]$,

$$\mathbf{E}|U_s^t - X_s|^2 + \mathbf{E}|V_s^t - Y_s|^2 \leqslant \alpha''\mathbf{E}\int_t^T [|U_s^t - X_s|^2 + |V_s^t - Y_s|^2]\,\mathrm{d}s.$$

So, for a small enough $T-t$,

$$\sup_{t\leqslant s\leqslant T}\mathbf{E}|U_s^t - X_s|^2 = \sup_{t\leqslant s\leqslant T}\mathbf{E}|V_s^t - Y_s|^2 = \mathbf{E}\int_t^T |W_s^t - Z_s|^2\,\mathrm{d}s = 0.$$

And, using continuity of the processes $(X_s)_{t\leqslant s\leqslant T}, (Y_s)_{t\leqslant s\leqslant T}, (U_s^t)_{t\leqslant s\leqslant T}, (V_s^t)_{t\leqslant s\leqslant T}$,

$$\mathbf{E}\sup_{t\leqslant s\leqslant T}|U_s^t - X_s|^2 = \mathbf{E}\sup_{t\leqslant s\leqslant T}|V_s^t - Y_s|^2 = \mathbf{E}\int_t^T |W_s^t - Z_s|^2\,\mathrm{d}s = 0.$$

This proves Lemma 3.3. □

Let us go back to the proof of Theorem 3.2. Let us denote $\tau_1 = T - \delta$. Then, from Lemma 3.3 and Theorem 3.1,

$$\forall s \in [\tau_1, T], \quad Y_s = \theta(s, X_s).$$

Noting that the map $\theta(\tau_1, .)$ is C_1-lipschitzian, and following the scheme developed in Section 2, we complete the proof with an induction. □

Remark 3.4. Note that one of the main point of the proof of Theorem 3.2 is the a priori estimate $\mathbf{P}\{\sup_{0\leqslant t\leqslant T}|Y_t| \leqslant C_0\} = 1$. Actually, such an estimate can be established under weaker assumptions than Assumption (A3). Indeed, assume that g satisfies

(A3.6): $\forall (t,x,y,z) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$,

$$\langle y, g(t,x,y,z)\rangle \leqslant \Lambda(0)(1+|y|^2+|y||z|),$$

instead of

$$\forall (t,x,y,z) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}, \quad |g(t,x,y,z)| \leqslant \Lambda(0)(1+|y|+|z|).$$

Then, following the proof of (2.1.7), every triple (X,Y,Z), which satisfies both problem (E) and the property $\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t|^2<\infty$, also satisfies $\mathbf{P}\{\sup_{0\leqslant t\leqslant T}|Y_t|\leqslant C_0\}=1$ (we recall that $\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t|^2<\infty$ is required to establish that $\mathbf{E}\int_t^T\langle Y_s, Z_s\,\mathrm{d}B_s\rangle=0$, see the proof of Theorem 1.1).

Hence, if (f,g,h,σ) satisfies (A3.6) and (A3) with the second line of (A3.2) replaced by

$$\forall (t,x,y,z) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}, \quad |g(t,x,y,z)| \leqslant \Lambda(|y|)(1+|z|),$$

then for every $\mathscr{G}_0$ measurable random vector ξ with finite second moment, there exists a unique triple (X,Y,Z) satisfying both

$$\mathbf{E}\int_0^T (|X_t|^2+|Z_t|^2)\,\mathrm{d}t+\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t|^2<\infty,$$

and problem (E).

3.2. Extension of the uniqueness property

We now give under some appropriate assumptions a stronger uniqueness result which is actually well adapted to convergence in law problems (see for example the article of Pardoux (1999) related to BSDEs case and to the application to the homogenization of semi-linear parabolic PDEs).

Assumption (A3′). We say that the functions f,F,g,G,h and σ satisfy Assumption (A3′) if there exist three non-decreasing functions $k,K,\Lambda:\mathbf{R}^+\to\mathbf{R}^+$, and a non-increasing function $\lambda:\mathbf{R}^+\to\mathbf{R}^+\setminus\{0\}$, such that they satisfy the following properties:

(A3′.0): $F:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to(\mathbf{R}^{Q\times P})^P$,

$G:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to(\mathbf{R}^{Q\times P})^Q$,

$f:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to\mathbf{R}^P$,

$g:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to\mathbf{R}^Q$,

$\sigma:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to\mathbf{R}^{P\times P}$,

$h:\mathbf{R}^P\to\mathbf{R}^Q$.

(A3′.1): $\forall t \in [0,T],\ \forall (x,y) \in \mathbf{R}^P \times \mathbf{R}^Q,\ \forall (x',y') \in \mathbf{R}^P \times \mathbf{R}^Q,$

$$|F(t,x,y) - F(t,x,y')| + |f(t,x,y) - f(t,x,y')| \leqslant K(|y| + |y'|)|y - y'|,$$

$$|G(t,x,y) - G(t,x',y)| + |g(t,x,y) - g(t,x',y)| \leqslant K(|y|)|x - x'|,$$

$$|h(x) - h(x')| \leqslant k(0)|x - x'|,$$

$$|\sigma(t,x,y) - \sigma(t,x',y')|^2 \leqslant k^2(|y| + |y'|)(|x - x'|^2 + |y - y'|^2).$$

(A3′.2): $\forall t \in [0,T],\ \forall (x,y) \in \mathbf{R}^P \times \mathbf{R}^Q,\ \forall (x',y') \in \mathbf{R}^P \times \mathbf{R}^Q,\ \forall u \in \mathbf{R},\ \forall w \in \mathbf{R}^{Q\times P},$

$$\langle x - x', u(f(t,x,y) - f(t,x',y)) + (F(t,x,y) - F(t,x',y))w\rangle \leqslant K(|y|)(|u| + |w|)|x - x'|^2,$$

$$\langle y - y', u(g(t,x,y) - g(t,x,y')) + (G(t,x,y) - G(t,x,y'))w\rangle \leqslant K(|y|)(|u| + |w|)|y - y'|^2.$$

(A3′.3): $\forall (t,x,y) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q,$

$$|G(t,x,y)| + |g(t,x,y)| \leqslant \Lambda(0)(1 + |y|),$$

$$|F(t,x,y)| + |f(t,x,y)| + |\sigma(t,x,y)| + |h(x)| \leqslant \Lambda(|y|).$$

(A3′.4): $\forall (t,x,y) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q,$

$$\forall \zeta \in \mathbf{R}^P, \quad \langle \zeta, \sigma\sigma^*(t,x,y)\zeta\rangle \geqslant \lambda(|y|)|\zeta|^2.$$

(A3′.5): $\forall (t,x,y) \in [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q$, the functions $u \mapsto F(t,u,y)$, $u \mapsto f(t,u,y)$, $v \mapsto G(t,x,v)$ and $v \mapsto g(t,x,v)$ are continuous. Moreover, σ is continuous on its definition set.

Theorem 3.5 (Extension of uniqueness). *Suppose that* (A3′) *is in force. Suppose that there exist a $\mathscr{G}_0$-measurable random vector ξ with finite second moment and a $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$-progressively measurable process $(X_t, Y_t, M_t, m_t^1, m_t^2)_{0\leqslant t\leqslant T}$, with values in $\mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^Q \times \mathbf{R}^{P\times Q} \times \mathbf{R}^{Q\times Q}$, such that*

$$\begin{cases} \forall t \in [0,T], \\ X_t = \xi + \displaystyle\int_0^t f(s,X_s,Y_s)\,\mathrm{d}s + \mathrm{trace}([m^1,M]_t) + \int_0^t \sigma(s,X_s,Y_s)\,\mathrm{d}B_s, \\ Y_t = h(X_T) + \displaystyle\int_t^T g(s,X_s,Y_s)\,\mathrm{d}s + \mathrm{trace}([m^2,M]_T - [m^2,M]_t) - (M_T - M_t), \\ \mathbf{E}\displaystyle\int_0^T (|X_t|^2 + |Y_t|^2)\,\mathrm{d}t < \infty, \end{cases} \tag{3.5.1}$$

where $(M_t)_{0\leqslant t\leqslant T}$ *is a square integrable* $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$*-martingale (a priori non-continuous), satisfying* $M_0=0$, *and* $(m_t^1)_{0\leqslant t\leqslant T}$ *and* $(m_t^2)_{0\leqslant t\leqslant T}$ *are two continuous and square integrable* $\{\mathscr{G}_t\}_{0\leqslant t\leqslant T}$*-martingales satisfying*:

(3.5.2) $\forall p\in\{1,..,P\}$, $\forall q\in\{1,..,Q\}$, $t\mapsto[(m^1)_{p,q}]_t$ *is absolutely continuous and satisfies*:
- (i) $\forall p\in\{1,..,P\}$, $\sum_{q=1}^{Q}|\frac{\mathrm{d}[(m^1)_{p,q}]_r}{\mathrm{d}r}|\leqslant\Lambda(0)\,\mathrm{d}r\otimes\mathrm{d}\mathbf{P}$ a.e.
- (ii) $\forall(p,q,p')\in\{1,..,P\}\times\{1,..,Q\}\times\{1,..,P\}$, $[(m^1)_{p,q},(B)_{p'}]_t=\int_0^t F_{p,q,p'}(s,X_s,Y_s)\,\mathrm{d}s$.

(3.5.3) $\forall q\in\{1,..,Q\}$, $\forall q'\in\{1,..,Q\}$, $t\mapsto[(m^2)_{q,q'}]_t$ *is absolutely continuous and satisfies*:
- (i) $\forall q\in\{1,..,Q\}$, $\sum_{q'=1}^{Q}|\frac{\mathrm{d}[(m^2)_{q,q'}]_r}{\mathrm{d}r}|\leqslant\Lambda(0)\,\mathrm{d}r\otimes\mathrm{d}P$ a.e.
- (ii) $\forall(q,q',p)\in\{1,..,Q\}\times\{1,..,Q\}\times\{1,..,P\}$, $[(m^2)_{q,q'},(B)_p]_t=\int_0^t G_{q,q',p}(s,X_s,Y_s)\,\mathrm{d}s$.

In addition, note that $\forall t\in[0,T]$, $\mathrm{trace}([m^1,M]_t)$ *and* $\mathrm{trace}([m^2,M]_t)$ *stand respectively for the* $\mathbf{R}^P$ *and* $\mathbf{R}^Q$ *valued vectors defined by*

$$\forall p\in\{1,..,P\},\quad(\mathrm{trace}([m^1,M]_t))_p=\sum_{q=1}^{Q}[(m^1)_{p,q},(M)_q]_t,$$

$$\forall q\in\{1,..,Q\},\quad(\mathrm{trace}([m^2,M]_t))_q=\sum_{q'=1}^{Q}[(m^2)_{q,q'},(M)_{q'}]_t,$$

where $[\,,]$ *stands for the quadratic covariation.*

Then, the processes $(X_t)_{0\leqslant t\leqslant T}$, $(Y_t)_{0\leqslant t\leqslant T}$ *and* $(M_t)_{0\leqslant t\leqslant T}$ *are continuous and satisfy*

$$\mathbf{E}\sup_{0\leqslant t\leqslant T}|X_t-U_t^{0,\xi}|^2=\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t-V_t^{0,\xi}|^2=\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|M_t-\int_0^t W_s^{0,\xi}\,\mathrm{d}B_s\right|^2=0,$$

where for every $t\in[0,T]$, *and for every* $\mathscr{G}_t$*-measurable random vector* $\xi^{(t)}$, *satisfying* $\mathbf{E}|\xi^{(t)}|^2<\infty$, $(U_s^{t,\xi^{(t)}},V_s^{t,\xi^{(t)}},W_s^{t,\xi^{(t)}})_{t\leqslant s\leqslant T}$ *is the solution of*

$$\begin{cases}\forall s\in[t,T],\\ U_s=\xi^{(t)}+\displaystyle\int_t^s f(r,U_r,V_r)\,\mathrm{d}r+\int_t^s F(r,U_r,V_r)W_r\,\mathrm{d}r+\int_t^s\sigma(r,U_r,V_r)\,\mathrm{d}B_r,\\ V_s=h(U_T)+\displaystyle\int_s^T g(r,U_r,V_r)\,\mathrm{d}r+\int_s^T G(r,U_r,V_r)W_r\,\mathrm{d}r-\int_s^T W_r\,\mathrm{d}B_r,\\ \mathbf{E}\displaystyle\int_t^T(|U_r|^2+|V_r|^2+|W_r|^2)\,\mathrm{d}r<\infty.\end{cases}\tag{3.5.4}$$

Notations. From Theorems 3.1 and 3.2, we know that there exist a constant C_0 only depending on Λ and T, and a constant C_1, only depending on k, Λ, λ, P, Q and T, such that the map

$$\theta:[0,T]\times\mathbf{R}^P\to\mathbf{R}^Q,\quad (t,x)\mapsto V_t^{t,x},$$

satisfies

$$\forall (t,x)\in[0,T]\times\mathbf{R}^P,\quad |\theta(t,x)|\leqslant C_0,$$

$$\forall t\in[0,T],\quad \forall (x,x')\in(\mathbf{R}^P)^2,\quad |\theta(t,x')-\theta(t,x)|\leqslant C_1|x'-x|.$$

Moreover, for every $t\in[0,T]$, for every $\mathscr{G}_t$-measurable random vector $\xi^{(t)}$ with finite second moment, $(U_s^{t,\xi^{(t)}},V_s^{t,\xi^{(t)}},W_s^{t,\xi^{(t)}})_{t\leqslant s\leqslant T}$ satisfies

$$\mathbf{P}\{\forall s\in[t,T],\ V_s^{t,\xi^{(t)}}=\theta(s,U_s^{t,\xi^{(t)}})\}=1,$$

and

$$\mathbf{P}\otimes\mu\{(s,\omega)\in[t,T]\times\Omega,\ |W_s^{t,\xi^{(t)}}(\omega)|>C_2\}=0,$$

where C_2 only depends on k, λ, Λ, P, Q and T.

Remark. Under assumptions of Theorem 3.5, we note that

(i) $(Y_t)_{0\leqslant t\leqslant T}$ and $(M_t)_{0\leqslant t\leqslant T}$ are càd-làg.
(ii) From Kunita–Watanabe and Doob's maximal inequalities, we deduce $\mathbf{E}\sup_{0\leqslant t\leqslant T}|Y_t|^2<\infty$.
(iii) Following the proof of (2.1.7) and modifying C_0 if necessary, $\mathbf{P}\{\sup_{0\leqslant t\leqslant T}|Y_t|^2\leqslant C_0\}=1$.
(iv) $(X_t)_{0\leqslant t\leqslant T}$ is continuous and $\mathbf{E}\sup_{0\leqslant t\leqslant T}|X_t|^2<\infty$.

Let us prove Theorem 3.5.

Proof. Let us consider $(X_s,Y_s,M_s,m_s^1,m_s^2)_{0\leqslant s\leqslant T}$ given by the statement. Then, for every $t\in[0,T]$, $(X_s,Y_s,M_s,m_s^1,m_s^2)_{t\leqslant s\leqslant T}$ satisfies

$$\forall s\in[t,T],$$

$$X_s=X_t+\int_t^s f(u,X_u,Y_u)\,\mathrm{d}u+\mathrm{trace}([m^1,M]_s-[m^1,M]_t)+\int_t^s\sigma(u,X_u,Y_u)\,\mathrm{d}B_u,$$

$$Y_s=h(X_T)+\int_s^T g(u,X_u,Y_u)\,\mathrm{d}u+\mathrm{trace}([m^2,M]_T-[m^2,M]_s)-(M_T-M_s),$$

$$\mathbf{E}\int_t^T(|X_u|^2+|Y_u|^2)\,\mathrm{d}u<\infty.$$

The proof is based on the following lemma, which is very close to Lemma 3.3.

Lemma 3.6. *There exists a real $\delta'>0$, only depending on Λ, K, k, P, Q, C_0, C_1 and C_2, such that for every $t\in[T-\delta',T]$, the processes $(X_s)_{t\leqslant s\leqslant T}$, $(Y_s)_{t\leqslant s\leqslant T}$ and*

$(M_s)_{t \leqslant s \leqslant T}$ *are continuous and satisfy*

$$\mathbf{E} \sup_{t \leqslant s \leqslant T} |X_s - U_s^t|^2 = \mathbf{E} \sup_{t \leqslant s \leqslant T} |Y_s - V_s^t|^2$$

$$= \mathbf{E}[\mathrm{trace}([M - N^t]_T) - \mathrm{trace}([M - N^t]_t)] = 0;$$

where for every $t \in [0, T]$, $(U_s^t, V_s^t, W_s^t)_{t \leqslant s \leqslant T}$ *is the solution of*

$$\begin{cases} \forall s \in [t, T], \\ U_s^t = X_t + \int_t^s f(r, U_r^t, V_r^t)\,\mathrm{d}r + \int_t^s F(r, U_r^t, V_r^t) W_r^t\,\mathrm{d}r + \int_t^s \sigma(r, U_r^t, V_r^t)\,\mathrm{d}B_r, \\ V_s^t = h(U_T^t) + \int_s^T g(r, U_r^t, V_r^t)\,\mathrm{d}r + \int_s^T G(r, U_r^t, V_r^t) W_r^t\,\mathrm{d}r - \int_s^T W_r^t\,\mathrm{d}B_r, \\ \mathbf{E}\int_t^T (|U_r^t|^2 + |V_r^t|^2 + |W_r^t|^2)\,\mathrm{d}r < \infty, \end{cases}$$

and $\forall t \in [0, T]$, $(N_s^t)_{t \leqslant s \leqslant T}$ *is defined by*

$$\forall s \in [t, T], \quad N_s^t = \int_t^s W_u^t\,\mathrm{d}B_u.$$

Proof. Let us consider $t \in [0, T]$ and $(U_s^t, V_s^t, W_s^t)_{t \leqslant s \leqslant T}$ defined by the statement of Lemma 3.6. Moreover, we define $\forall s \in [t, T]$,

$$\forall (p, q) \in \{1, .., P\} \times \{1, .., Q\}, \quad (n_s^{1,t})_{p,q} = \int_t^s \langle F_{p,q,.}(r, U_r^t, V_r^t), \mathrm{d}B_r \rangle,$$

$$\forall (q, q') \in \{1, .., Q\} \times \{1, .., Q\}, \quad (n_s^{2,t})_{q,q'} = \int_t^s \langle G_{q,q',.}(r, U_r^t, V_r^t), \mathrm{d}B_r \rangle.$$

So, according to the adopted notations, $\forall s \in [t, T]$,

$$\mathrm{trace}([n^{1,t}, N^t]_s) = \int_t^s F(r, U_r^t, V_r^t) W_r^t\,\mathrm{d}r,$$

$$\mathrm{trace}([n^{2,t}, N^t]_s) = \int_t^s G(r, U_r^t, V_r^t) W_r^t\,\mathrm{d}r.$$

Hence, from Itô's formula, $\forall s \in [t, T]$,

$$\mathbf{E}|U_s^t - X_s|^2 = 2\mathbf{E}\int_t^s \langle U_r^t - X_r, f(r, U_r^t, V_r^t) - f(r, X_r, Y_r) \rangle\,\mathrm{d}r$$

$$+ 2\mathbf{E}\int_t^s \langle U_r^t - X_r, \mathrm{d}[\mathrm{trace}([n^{1,t}, N^t]_r - [m^1, M^t]_r)] \rangle$$

$$+ \mathbf{E}\int_t^s |\sigma(r, U_r^t, V_r^t) - \sigma(r, X_r, Y_r)|^2\,\mathrm{d}r.$$

Now, using (3.5.2), we have $\forall r \in [t,T]$,

$$\mathrm{trace}([n^{1,t} - m^1, N^t]_r) = \int_t^r (F(u, U_u^t, V_u^t) - F(u, X_u, Y_u)) W_u^t \, \mathrm{d}u.$$

Moreover, using Kunita–Watanabe inequality, $\forall s \in [t,T]$,

$$\mathbf{E} \int_t^s \langle X_r - U_r^t, \mathrm{d}\, \mathrm{trace}([m^1, N^t - M^t]_r) \rangle$$

$$\leqslant \sum_{l=1}^{P} \sum_{q=1}^{Q} \mathbf{E} \left[\left(\int_t^s ((X_r)_l - (U_r^t)_l)^2 \, \mathrm{d}[(m^1)_{l,q}]_r \right)^{1/2} \right.$$

$$\left. ([(N^t - M^t)_q]_s - [(N^t - M^t)_q]_t)^{1/2} \right].$$

Therefore, using Assumption (A3′) as well as property (3.5.2), and following the proof of Lemma 3.3, we can find a constant β only depending on k, K, Λ, P, Q, C_0, C_1 and C_2, such that $\forall s \in [t,T]$,

$$\mathbf{E}|U_s^t - X_s|^2 \leqslant \beta \mathbf{E} \int_t^s (|U_r^t - X_r|^2 + |V_r^t - Y_r|^2) \, \mathrm{d}r$$

$$+ \frac{1}{4} \frac{1}{C_1^2} \mathbf{E}[\mathrm{trace}([M - N^t]_s) - \mathrm{trace}([M - N^t]_t)]. \tag{3.6.1}$$

Moreover, Itô's formula also shows that $\forall s \in [t,T]$,

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}[\mathrm{trace}([M - N^t]_T) - \mathrm{trace}([M - N^t]_s)]$$

$$= \mathbf{E}|h(X_T) - h(U_T^t)|^2 + 2\mathbf{E} \int_s^T \langle Y_{r-} - V_r^t, g(r, X_r, Y_r) - g(r, U_r^t, V_r^t) \rangle \, \mathrm{d}r$$

$$+ 2\mathbf{E} \int_s^T \langle Y_{r-} - V_r^t, \mathrm{d}\, \mathrm{trace}([n^{2,t}, N^t]_r - [m^2, M]_r) \rangle.$$

In addition, we have $\forall r \in [t,T]$,

$$\mathrm{trace}([n^{2,t} - m^2, N^t]_r) = \int_t^r (G(u, U_u^t, V_u^t) - G(u, X_u, Y_u)) W_u^t \, \mathrm{d}u.$$

Therefore, considering (3.5.3) and using once again Kunita–Watanabe inequality, we show from (3.6.1) that there exists a constant β' only depending on K, k, Λ, P, Q, C_0, C_1 and C_2, such that $\forall s \in [t,T]$,

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}[\mathrm{trace}([M - N^t]_T) - \mathrm{trace}([M - N^t]_t)]$$

$$\leqslant \beta' \, \mathbf{E} \int_t^T (|U_r^t - X_r|^2 + |V_r^t - Y_r|^2 + |V_r^t - Y_{r-}|^2) \, \mathrm{d}r.$$

So, noting that

$$\mathbf{E} \int_t^T |V_r^t - Y_r|^2 \, \mathrm{d}r = \mathbf{E} \int_t^T |V_r^t - Y_{r-}|^2 \, \mathrm{d}r,$$

we have $\forall s \in [t,T]$,

$$\mathbf{E}|Y_s - V_s^t|^2 + \mathbf{E}[\mathrm{trace}([M - N^t]_T) - \mathrm{trace}([M - N^t]_t)]$$
$$\leqslant \beta' \, \mathbf{E} \int_t^T (|U_r^t - X_r|^2 \, \mathrm{d}r + |V_r^t - Y_r|^2) \, \mathrm{d}r. \tag{3.6.2}$$

Eventually, injecting (3.6.2) in (3.6.1), there exists a constant β'' only depending on k, K, Λ, P, Q, C_0, C_1 and C_2 such that $\forall s \in [t,T]$,

$$\mathbf{E}(|U_s^t - X_s|^2 + |V_s^t - Y_s|^2) \leqslant \beta'' \, \mathbf{E} \int_t^T [|U_s^t - X_s|^2 + |V_s^t - Y_s|^2] \, \mathrm{d}s.$$

Hence, for a small enough $T - t$,

$$\sup_{t \leqslant s \leqslant T} \mathbf{E}|U_s^t - X_s|^2 = \sup_{t \leqslant s \leqslant T} \mathbf{E}|V_s^t - Y_s|^2$$
$$= \mathbf{E}[\mathrm{trace}([M - N^t]_T) - \mathrm{trace}([M - N^t]_t)] = 0.$$

Then, using the right continuity of the processes $(X_s)_{t \leqslant s \leqslant T}$, $(Y_s)_{t \leqslant s \leqslant T}$, $(U_s^t)_{t \leqslant s \leqslant T}$, $(V_s^t)_{t \leqslant s \leqslant T}$, we prove that

$$\mathbf{E} \sup_{t \leqslant s \leqslant T} |U_s^t - X_s|^2 = \mathbf{E} \sup_{t \leqslant s \leqslant T} |V_s^t - Y_s|^2 = 0.$$

This proves that $(X_s)_{t \leqslant s \leqslant T}$, $(Y_s)_{t \leqslant s \leqslant T}$ and $(M_s)_{t \leqslant s \leqslant T}$ are continuous. $\square$

Let us go back to the proof of Theorem 3.5. We let $\tau_1' = T - \delta'$. So, from Theorem 3.1:

$$\forall s \in [\tau_1', T], \quad Y_s = \theta(s, X_s).$$

Using the same scheme as in the proof of Theorem 3.2, and noting that $\mathrm{trace}([M - N^0]_0) = 0$, we complete the proof with an induction. $\square$

Acknowledgements

The author wishes to thank Étienne Pardoux for his help and support during the preparation of this paper.

Appendix A. Random coefficients case

The goal of the appendices is to present a probabilistic method to obtain some regularity results on the map θ given by Corollary 1.5 and in particular to prove that, under appropriate assumptions and over a small enough time duration T, it is a classical solution of the system (E′). Actually, to show such properties, it may be more convenient to extend first the results of the part one to the case of random coefficients. This is what we do in Appendix A.

However, the proofs being almost the same as in the case of deterministic coefficients, we just give the statements of the results without detailing the proofs.

Assumption (A.A1). We say that the functions f, g, h and σ satisfy Assumption (A.A1) if there exist three constants $K, \Lambda, q \geqslant 1$, such that:

(A.A1.0): $f:\Omega \times [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P} \to \mathbf{R}^P$,

$$g:\Omega \times [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P} \to \mathbf{R}^Q,$$

$$\sigma:\Omega \times [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \to \mathbf{R}^{P\times P},$$

$$h:\Omega \times \mathbf{R}^P \to \mathbf{R}^Q,$$

are respectively $\mathscr{P}^0 \otimes \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{R}^Q) \otimes \mathscr{B}(\mathbf{R}^{Q\times P})/\mathscr{B}(\mathbf{R}^P)$, $\mathscr{P}^0 \otimes \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{R}^Q) \otimes \mathscr{B}(\mathbf{R}^{Q\times P})/\mathscr{B}(\mathbf{R}^Q)$, $\mathscr{P}^0 \otimes \mathscr{B}(\mathbf{R}^P) \otimes \mathscr{B}(\mathbf{R}^Q)/\mathscr{B}(\mathbf{R}^{P\times P})$ and $\mathscr{G}^0_T \otimes \mathscr{B}(\mathbf{R}^P)/\mathscr{B}(\mathbf{R}^P)$ measurable, where $\mathscr{P}^0$ stands for the progressive σ-field with respect to the filtration $\{\mathscr{G}^0_t\}$, defined by

$$\forall t \in [0,T], \quad \mathscr{G}^0_t = \mathscr{G}^0 \vee \mathscr{F}_t.$$

(A.A1.1): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, $\forall (x',y',z') \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, and a.s.,

$$|f(t,x,y,z) - f(t,x,y',z')| \leqslant K(|y-y'| + |z-z'|),$$

$$|g(t,x,y,z) - g(t,x,y',z')| \leqslant K(|x-x'| + |z-z'|),$$

$$|h(x) - h(x')| \leqslant K|x-x'|,$$

$$|\sigma(t,x,y) - \sigma(t,x',y')|^2 \leqslant K^2(|x-x'|^2 + |y-y'|^2).$$

(A.A1.2): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, $\forall (x',y') \in \mathbf{R}^P \times \mathbf{R}^Q$, and a.s.,

$$\langle x-x',\ f(t,x,y,z) - f(t,x',y,z)\rangle \leqslant K|x-x'|^2,$$

$$\langle y-y',\ g(t,x,y,z) - g(t,x,y',z)\rangle \leqslant K|y-y'|^2.$$

(A.A1.3): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$, and a.s.,

$$|f(t,x,y,z)| \leqslant \Lambda(|f(t,0,0,0)| + |x| + |y| + |z|),$$

$$|g(t,x,y,z)| \leqslant \Lambda(|g(t,0,0,0)| + |x| + |y| + |z|).$$

(A.A1.4): The following holds:

(i) $\mathbf{E}[|h(0)|^2 + \int_0^T (|f(s,0,0,0)|^2 + |g(s,0,0,0)|^2)\,\mathrm{d}s + \int_0^T |\sigma(s,0,0)|^2\,\mathrm{d}s] < \infty$,

(ii) $\mathbf{E}(\Pi_0)^q < \infty$,

where $\forall t \in [0,T]$,

$$\Pi_t = |h(0)|^2 + \left(\int_t^T (|f(s,0,0,0)| + |g(s,0,0,0)|)\,\mathrm{d}s\right)^2 + \int_t^T |\sigma(s,0,0)|^2\,\mathrm{d}s.$$

Note that we also define (it will be used next) for $0 \leqslant s \leqslant t \leqslant T$,

$$\Pi_{s,t} = \left(\int_s^t (|f(u,0,0,0)| + |g(u,0,0,0)|)\,\mathrm{d}u \right)^2 + \int_s^t |\sigma(u,0,0)|^2\,\mathrm{d}u.$$

(A.A1.5): $\forall t \in [0,T]$, $\forall (x,y,z) \in \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$ and a.s., the functions $u \mapsto f(t,u,y,z)$ and $v \mapsto g(t,x,v,z)$ are continuous.

Theorem A.1 (Existence and uniqueness in small time duration). *Assume that* (A.A1) *is in force. Then, every solution* $(X_t, Y_t, Z_t)_{0\leqslant t\leqslant T}$ *of the problem* (E) *satisfies*:

(i) $(X_t)_{0\leqslant t\leqslant T}$ *and* $(Y_t)_{0\leqslant t\leqslant T}$ *are continuous* $\{\mathscr{G}_t^0\}_{0\leqslant t\leqslant T}$*-adapted.*
(ii) $\mathbf{E}(\sup_{0\leqslant t\leqslant T}|X_t|^2 + \sup_{0\leqslant t\leqslant T}|Y_t|^2) < \infty$.

Moreover, there exists a constant $\tilde{C}_K^{(1)} > 0$, *only depending on* K, *such that for every* $\mathscr{G}_0$*-measurable random vector* ξ, *satisfying* $\mathbf{E}|\xi|^2 < \infty$, *and for every* $T \leqslant \tilde{C}_K^{(1)}$, *the problem* (E) *admits a unique solution.*

Theorem A.2 (Main estimate). *Suppose that* (A.A1) *is in force. Then, there exists a constant* $0 < \tilde{C}_K^{(2)} \leqslant \tilde{C}_K^{(1)}$, *only depending on* K, *such that for every* $T \leqslant \tilde{C}_K^{(2)}$, *for every vector of functions* $(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\sigma})$ *satisfying Assumption* (A.A1) *with respect to* K *and* Λ, *for every* $A \in \mathscr{G}_0$, *and for all* $\mathscr{G}_0$*-measurable random vectors* ξ *and* $\tilde{\xi}$ *with finite second moment, we have the following estimate*:

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|X_s - \tilde{X}_s|^2\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|Y_s - \tilde{Y}_s|^2\right) \\
&\qquad + \mathbf{E}\int_0^T (\mathbf{1}_A|Z_s - \tilde{Z}_s|^2)\,\mathrm{d}s \\
&\quad \leqslant \tilde{c}_K^{(2)} \Bigg[\mathbf{E}(\mathbf{1}_A|\xi - \tilde{\xi}|^2) + \mathbf{E}(\mathbf{1}_A|(h - \tilde{h})(X_T)|^2) \\
&\qquad + \mathbf{E}\left(\mathbf{1}_A \int_0^T (|f - \tilde{f}| + |g - \tilde{g}|)(r, X_r, Y_r, Z_r)\,\mathrm{d}r\right)^2 \\
&\qquad + \mathbf{E}\int_0^T (\mathbf{1}_A|\sigma - \tilde{\sigma}|^2)(r, X_r, Y_r)\,\mathrm{d}r \Bigg],
\end{aligned} \tag{A.2.1}$$

where $\tilde{c}_K^{(2)}$ *only depends on* K, *and where the processes* $(X_s, Y_s, Z_s)_{0\leqslant s\leqslant T}$ *and* $(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)_{0\leqslant s\leqslant T}$ *stand for the solutions of the problems associated to the coefficients* (f, g, h, σ) *and* $(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{\sigma})$ *and to the initial conditions* $(0, \xi)$ *and* $(0, \tilde{\xi})$.

Corollary A.3 (Dependence upon initial conditions). *Suppose that* (A.A1) *is in force. For every* $T \leqslant \tilde{C}_K^{(2)}$, *for every* $t \in [0,T]$, *and for every* $\mathscr{G}_t^0$*-measurable random vector*

ξ with finite second moment, we denote by $(X_s^{t,\xi}, Y_s^{t,\xi}, Z_s^{t,\xi})_{t\leqslant s\leqslant T}$ *the unique solution of the problem*

$$\begin{cases} \forall s\in[t,T], \\ X_s = \xi + \displaystyle\int_t^s f(r,X_r,Y_r,Z_r)\,\mathrm{d}r + \int_t^s \sigma(r,X_r,Y_r)\,\mathrm{d}B_r, \\ Y_s = h(X_T) + \displaystyle\int_s^T g(r,X_r,Y_r,Z_r)\,\mathrm{d}r - \int_s^T Z_r\,\mathrm{d}B_r, \\ \mathbf{E}\displaystyle\int_t^T (|X_s|^2+|Y_s|^2+|Z_s|^2)\,\mathrm{d}s<\infty. \end{cases}$$

If $\xi=x$ a.s., $x\in\mathbf{R}^P$, *this one can be extended to the whole interval* $[0,T]$ *by putting*

$$\begin{cases} \forall 0\leqslant s\leqslant t, \quad X_s^{t,x}=x, \quad Y_s^{t,x}=\mathbf{E}(Y_t^{t,x}\,|\,\mathscr{G}_s), \\ Y_t^{t,x} = Y_0^{t,x} + \displaystyle\int_0^t Z_u^{t,x}\,\mathrm{d}B_u. \end{cases}$$

Then, the following properties are satisfied:

(A.3.1) $\forall(t,x)\in[0,T]\times\mathbf{R}^P,\ \forall A\in\mathscr{G}_t^0,$

$$\mathbf{E}\left(\mathbf{1}_A\sup_{t\leqslant s\leqslant T}|X_s^{t,x}|^2\right)+\mathbf{E}\left(\mathbf{1}_A\sup_{t\leqslant s\leqslant T}|Y_s^{t,x}|^2\right)+\mathbf{E}\left(\mathbf{1}_A\int_t^T|Z_s^{t,x}|^2\,\mathrm{d}s\right)$$
$$\leqslant \tilde{c}_K^{(2)}\,\mathbf{E}(\mathbf{1}_A(|x|^2+\Pi_t)).$$

(A.3.2) *There exists a constant* $\tilde{c}_{K,\Lambda}^{(2)}$, *only depending on* K *and* Λ, *such that* $\forall(t,x)\in[0,T]\times\mathbf{R}^P,\ \forall(t',x')\in[0,T]\times\mathbf{R}^P,\ t\leqslant t'$ *and* $\forall A\in\mathscr{G}_t^0,$

$$\mathbf{E}\left(\mathbf{1}_A\sup_{t\leqslant s\leqslant T}|X_s^{t',x'}-X_s^{t,x}|^2\right)+\mathbf{E}\left(\mathbf{1}_A\sup_{t\leqslant s\leqslant T}|Y_s^{t',x'}-Y_s^{t,x}|^2\right)$$
$$+\mathbf{E}\left(\mathbf{1}_A\int_t^T|Z_s^{t',x'}-Z_s^{t,x}|^2\,\mathrm{d}s\right)$$
$$\leqslant \tilde{c}_K^{(2)}\,\mathbf{E}(\mathbf{1}_A(|x-x'|^2+2\Pi_{t,t'}))+\tilde{c}_{K,\Lambda}^{(2)}(t'-t)\mathbf{E}(\mathbf{1}_A(|x|^2+\Pi_t)).$$

Corollary A.4. *Suppose that* (A.A1) *is in force and keep the notations of Corollary* A.3. *Then, for every* $T\leqslant\tilde{C}_K^{(2)}$, *for every* $t\in[0,T]$, *there exists a version denoted* $\theta(t,.)$ *of the process* $Y_t^{t,\cdot}:\Omega\times\mathbf{R}^P\to\mathbf{R}^Q$ *satisfying*:

(A.4.1) *For every* $t\in[0,T]$, $\theta(t,.)$ *is* $\mathscr{G}_t^0\otimes\mathscr{B}(\mathbf{R}^P)/\mathscr{B}(\mathbf{R}^Q)$ *measurable.*

(A.4.2) $\forall x\in\mathbf{R}^P,\quad |\theta(t,x)|^2\leqslant\tilde{c}_K^{(2)}(|x|^2+\mathbf{E}(\Pi_t\,|\,\mathscr{G}_t^0)).$

(A.4.3) $\forall (x,x') \in (\mathbf{R}^P)^2, \quad |\theta(t,x)-\theta(t,x')|^2 \leqslant \tilde{c}_K^{(2)}|x-x'|^2.$

(A.4.4) *For every $t \in [0,T]$, for every $\mathscr{G}_t^0$-measurable random vector ξ with finite second moment, for every $s \in [t,T]$,*

$$Y_s^{t,\xi} = \theta(s, X_s^{t,\xi}) \quad \text{a.s.}$$

Theorem A.5 (Higher order estimates). *Suppose that* (A.A1) *is in force. Then, for every $1 \leqslant p \leqslant q$, there exists a constant $\tilde{c}_{p,K}$, only depending on p and K such that for every $T \leqslant \tilde{C}_K^{(2)}$ and for every $\mathscr{G}_0$-measurable random vector ξ with finite $2p$*th *moment, the process $(X_s^{0,\xi}, Y_s^{0,\xi}, Z_s^{0,\xi})_{0\leqslant s\leqslant T}$ satisfies the following:*

$$\mathbf{E}\left(\sup_{0\leqslant s\leqslant T}|X_s^{0,\xi}|^{2p} + \sup_{0\leqslant s\leqslant T}|Y_s^{0,\xi}|^{2p}\right) + \mathbf{E}\left(\int_0^T |Z_s^{0,\xi}|^2\,\mathrm{d}s\right)^p \leqslant \tilde{c}_{p,K}(\mathbf{E}\,|\xi|^{2p} + \mathbf{E}\,(\Pi_0)^p). \tag{A.5.1}$$

For all vectors of functions (f,g,h,σ) and $(\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma})$ satisfying Assumption (A.A1) *with respect to the constants K, Λ and q, for all $\mathscr{G}_0$-measurable random vectors ξ and $\tilde{\xi}$ with finite $2p$*th *moment, and for every $A \in \mathscr{G}_0$, the following estimate holds:*

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|\tilde{X}_s - X_s|^{2p}\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^{2p}\right) \\
&\quad + \mathbf{E}\left[\mathbf{1}_A\left(\int_0^T |\tilde{Z}_s - Z_s|^2\,\mathrm{d}s\right)^p\right] \\
&\quad \leqslant \tilde{c}_{p,K}\,\mathbf{E}\left[\mathbf{1}_A\left(|\tilde{\xi}-\xi|^{2p} + |\tilde{h}-h|^{2p}(X_T)\right.\right. \\
&\qquad + \left(\int_0^T (|\tilde{f}-f| + |\tilde{g}-g|)(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p} \\
&\qquad \left.\left. + \left(\int_0^T |\tilde{\sigma}-\sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\right)^p\right)\right],
\end{aligned} \tag{A.5.2}$$

where $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$ and $(\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)_{0\leqslant s\leqslant T}$ stand for the solutions associated to the coefficients (f,g,h,σ) and $(\tilde{f},\tilde{g},\tilde{h},\tilde{\sigma})$ and to the initial conditions $(0,\xi)$ and $(0,\tilde{\xi})$.

Proof. We keep the notations given in the statement. We firstly assume that the processes $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$ and $(\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)_{0\leqslant s\leqslant T}$ satisfy the following assumptions:

$$(L_p) \quad \mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s|^{2p} + \mathbf{E}\left(\int_0^T |Z_s|^2\,\mathrm{d}s\right)^p < \infty,$$

and

$$(\tilde{L}_p) \quad \mathbf{E} \sup_{0 \leqslant s \leqslant T} |\tilde{X}_s|^{2p} + \mathbf{E} \sup_{0 \leqslant s \leqslant T} |\tilde{Y}_s|^{2p} + \mathbf{E}\left(\int_0^T |\tilde{Z}_s|^2 \, \mathrm{d}s \right)^p < \infty.$$

Let us then prove that there exists a constant $\tilde{C}_{p,K} > 0$, only depending on p and K, such that for every $T \leqslant \tilde{C}_{p,K}$, the processes $(X_s, Y_s, Z_s)_{0 \leqslant s \leqslant T}$ and $(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)_{0 \leqslant s \leqslant T}$ satisfy the inequality (A.5.2). To this end, we also assume that $T \leqslant 1$.

Using Itô's calculus, we have for every $\varepsilon > 0$:

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0 \leqslant s \leqslant T} (\varepsilon + |\tilde{X}_s - X_s|^2)^p \right) \\
&\quad \leqslant \mathbf{E}(\mathbf{1}_A(\varepsilon + |\tilde{\xi} - \xi|^2)^p) \\
&\qquad + 2p\, \mathbf{E} \sup_{0 \leqslant t \leqslant T} \left(\int_0^t \mathbf{1}_A(\varepsilon + |\tilde{X}_s - X_s|^2)^{p-1} \langle \tilde{X}_s - X_s, \tilde{f}(s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s) \right. \\
&\qquad \left. - \ f(s, X_s, Y_s, Z_s) \rangle \, \mathrm{d}s \right) + \ 2p\, \mathbf{E} \sup_{0 \leqslant t \leqslant T} \left(\int_0^t \mathbf{1}_A(\varepsilon + |\tilde{X}_s - X_s|^2)^{p-1} \right. \\
&\qquad \left. \times \langle \tilde{X}_s - X_s, (\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s) - \ \sigma(s, X_s, Y_s)) \, \mathrm{d}B_s \rangle \right) \\
&\qquad + p\, \mathbf{E} \int_0^t \mathbf{1}_A(\varepsilon + |\tilde{X}_s - X_s|^2)^{p-1} |\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s) - \sigma(s, X_s, Y_s)|^2 \, \mathrm{d}s \\
&\qquad + 2p(p-1) \sum_{j=1}^{P} \mathbf{E} \int_0^t \mathbf{1}_A(\varepsilon + |\tilde{X}_s - X_s|^2)^{p-2} \\
&\qquad \times \left(\sum_{i=1}^{P} (\tilde{X}_s - X_s)_i (\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s) - \sigma(s, X_s, Y_s))_{i,j} \right)^2 \mathrm{d}s.
\end{aligned}$$

Using Burkholder–Davis–Gundy inequalities as well as Assumption (A.A1), and letting $\varepsilon \to 0$, we show that there exists a constant $\tilde{c}_{p,K}$, only depending on p and K, such that

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0 \leqslant s \leqslant T} |\tilde{X}_s - X_s|^{2p} \right) \\
&\quad \leqslant \mathbf{E}(\mathbf{1}_A |\tilde{\xi} - \xi|^{2p}) + \ \tilde{c}_{p,K} \left[\mathbf{E} \int_0^T \mathbf{1}_A |\tilde{X}_s - X_s|^{2p-1} (|\tilde{X}_s - X_s| \right. \\
&\quad + |\tilde{Y}_s - Y_s| + |\tilde{Z}_s - Z_s|) \, \mathrm{d}s + \ \mathbf{E} \int_0^T \mathbf{1}_A |\tilde{X}_s - X_s|^{2p-1} |\tilde{f} - f|(s, X_s, Y_s, Z_s) \, \mathrm{d}s
\end{aligned}$$

$$+\mathbf{E}\int_0^T \mathbf{1}_A|\tilde{X}_s - X_s|^{2p-2}(|\tilde{X}_s - X_s|^2 + |\tilde{Y}_s - Y_s|^2)\,\mathrm{d}s$$

$$+\mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{X}_s - X_s|^{4p-2}|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s) - \sigma(s,X_s,Y_s)|^2\,\mathrm{d}s\right)^{1/2}$$

$$+\ \mathbf{E}\int_0^T \mathbf{1}_A|\tilde{X}_s - X_s|^{2p-2}|\tilde{\sigma} - \sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\Bigg].$$

Using Young's inequality and modifying $\tilde{c}_{p,K}$ if necessary, this shows that

$$\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|\tilde{X}_s - X_s|^{2p}\right)$$

$$\leqslant \mathbf{E}(\mathbf{1}_A|\tilde{\xi} - \xi|^{2p}) + \tilde{c}_{p,K}\Bigg[\mathbf{E}\int_0^T \mathbf{1}_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})\,\mathrm{d}s$$

$$+\mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{f} - f|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}$$

$$+\mathbf{E}\left(\int_0^T \mathbf{1}_A|\tilde{\sigma} - \sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\right)^{p}$$

$$+\ T^p\,\mathbf{E}\left(\int_0^T |\tilde{Z}_s - Z_s|^2\,\mathrm{d}s\right)^{p}\Bigg]. \tag{A.5.3}$$

On the other hand, using once again Itô's formula as well as Young's formula, and modifying $\tilde{c}_{p,K}$ if necessary, we have for every $t\in[0,T]$,

$$\mathbf{E}(\mathbf{1}_A|\tilde{Y}_t - Y_t|^{2p}) + \mathbf{E}\int_t^T \mathbf{1}_A|\tilde{Y}_s - Y_s|^{2p-2}|\tilde{Z}_s - Z_s|^2\,\mathrm{d}s$$

$$\leqslant \tilde{c}_{p,K}\Bigg[\mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^{2p}) + \mathbf{E}\int_t^T \mathbf{1}_A|\tilde{Y}_s - Y_s|^{2p-2}$$

$$\times\langle \tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - g(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\Bigg]. \tag{A.5.4}$$

Moreover, using Burkholder–Davis–Gundy inequalities and modifying $\tilde{c}_{p,K}$ if necessary, we obtain

$$\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^{2p}\right)$$

$$\leqslant \tilde{c}_{p,K}\Bigg[\mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^{2p}) + \mathbf{E}\sup_{0\leqslant t\leqslant T}\left(\int_t^T \mathbf{1}_A|\tilde{Y}_s - Y_s|^{2p-2}\right.$$

$$\times\langle\tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s) - g(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s\bigg)$$

$$+\ \mathbf{E}\left(\mathbf{1}_A\int_0^T|\tilde{Y}_s - Y_s|^{4p-2}|\tilde{Z}_s - Z_s|^2\,\mathrm{d}s\right)^{1/2}\Bigg].$$

Therefore, modifying $\tilde{c}_{p,K}$ if necessary, and using Young's estimate, we obtain

$$\mathbf{E}\left(\mathbf{1}_A\sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^{2p}\right) + \mathbf{E}\int_0^T\mathbf{1}_A|\tilde{Y}_s - Y_s|^{2p-2}|\tilde{Z}_s - Z_s|^2\,\mathrm{d}s$$

$$\leqslant \tilde{c}_{p,K}\left[\mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^{2p}) + \mathbf{E}\int_0^T\mathbf{1}_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})\,\mathrm{d}s\right.$$

$$\left.+\mathbf{E}\int_0^T\mathbf{1}_A|\tilde{Y}_s - Y_s|^{2p-1}|\tilde{Z}_s - Z_s|\,\mathrm{d}s + \mathbf{E}\left(\int_0^T\mathbf{1}_A|\tilde{g} - g|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}\right].$$

Hence, modifying $\tilde{c}_{p,K}$ if necessary,

$$\mathbf{E}\left(\mathbf{1}_A\sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^{2p}\right)$$

$$\leqslant \tilde{c}_{p,K}\left[\mathbf{E}(\mathbf{1}_A|\tilde{h}(\tilde{X}_T) - h(X_T)|^{2p}) + \mathbf{E}\int_0^T\mathbf{1}_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})\,\mathrm{d}s\right.$$

$$\left.+\ \mathbf{E}\left(\int_0^T\mathbf{1}_A|\tilde{g} - g|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}\right]. \tag{A.5.5}$$

Moreover, from the inequality

$$\int_0^T|\tilde{Z}_s - Z_s|^2\,\mathrm{d}s \leqslant \sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^2 + 2\int_0^T\langle\tilde{Y}_s - Y_s, \tilde{g}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)$$

$$-g(s,X_s,Y_s,Z_s)\rangle\,\mathrm{d}s - 2\int_0^T\langle\tilde{Y}_s - Y_s, (\tilde{Z}_s - Z_s)\,\mathrm{d}B_s\rangle,$$

we deduce, modifying once again $\tilde{c}_{p,K}$ if necessary,

$$\mathbf{E}\left(\int_0^T\mathbf{1}_A|\tilde{Z}_s - Z_s|^2\,\mathrm{d}s\right)^p$$

$$\leqslant \tilde{c}_{p,K}\left[\mathbf{E}\left(\mathbf{1}_A\left(\sup_{0\leqslant s\leqslant T}|\tilde{Y}_s - Y_s|^{2p} + \sup_{0\leqslant s\leqslant T}|\tilde{X}_s - X_s|^{2p}\right)\right)\right.$$

$$\left.+\mathbf{E}\left(\int_0^T\mathbf{1}_A|\tilde{g} - g|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}\right]. \tag{A.5.6}$$

Therefore, considering (A.5.3), (A.5.5) and (A.5.6), and modifying $\tilde{c}_{p,K} > 0$ if necessary, there exists a constant $\tilde{C}_{p,K}$, only depending on p and K, such that for $T \leqslant \tilde{C}_{p,K}$,

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T} |\tilde{X}_s - X_s|^{2p}\right) + \mathbf{E}\left(\mathbf{1}_A \sup_{0\leqslant s\leqslant T} |\tilde{Y}_s - Y_s|^{2p}\right) \\
&\quad + \mathbf{E}\left(\int_0^T \mathbf{1}_A |\tilde{Z}_s - Z_s|^2 \,\mathrm{d}s\right)^p \\
&\quad \leqslant \tilde{c}_{p,K}\, \mathbf{E}\Bigg[\mathbf{1}_A \Bigg(|\tilde{\xi} - \xi|^{2p} + |\tilde{h} - h|^{2p}(X_T) \\
&\qquad + \left(\int_0^T (|\tilde{f} - f| + |\tilde{g} - g|)(s, X_s, Y_s, Z_s)\,\mathrm{d}s\right)^{2p} \\
&\qquad + \left(\int_0^T |\tilde{\sigma} - \sigma|^2(s, X_s, Y_s)\,\mathrm{d}s\right)^p\Bigg)\Bigg]. \qquad \text{(A.5.7)}
\end{aligned}$$

We have proved that for every $T \leqslant \tilde{C}_{p,K}$ for all processes $(X_s, Y_s, Z_s)_{0\leqslant s\leqslant T}$ and $(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)_{0\leqslant s\leqslant T}$ satisfying the assumptions L_p and $\tilde{L}_p$, the inequality (A.5.2) is verified.

We do not suppose anymore that the process $(X_s, Y_s, Z_s)_{0\leqslant s\leqslant T}$ satisfies the assumption L_p. Let us prove that there exists a constant $\tilde{C}^*_{p,K} > 0$, only depending on p and K, such that for every $T \leqslant \tilde{C}^*_{p,K}$:

$$\mathbf{E} \sup_{0\leqslant s\leqslant T} |X_s|^{2p} + \mathbf{E} \sup_{0\leqslant s\leqslant T} |Y_s|^{2p} + \mathbf{E}\left(\int_0^T |Z_s|^2 \,\mathrm{d}s\right)^p < \infty.$$

We know, from the proof of Theorem A.1, that the iterated scheme:

$$\begin{cases}
\forall t \in [0, T], \\
X_t^{n+1} = \xi + \displaystyle\int_0^t f(s, X_s^{n+1}, Y_s^n, Z_s^n)\,\mathrm{d}s + \int_0^t \sigma(s, X_s^{n+1}, Y_s^n)\,\mathrm{d}B_s, \\
Y_t^{n+1} = h(X_T^{n+1}) + \displaystyle\int_t^T g(s, X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1})\,\mathrm{d}s - \int_t^T Z_s^{n+1}\,\mathrm{d}B_s,
\end{cases}$$

satisfies:

$$\mathbf{E} \sup_{0\leqslant s\leqslant T} |X_s^n - X_s|^2 + \mathbf{E} \sup_{0\leqslant s\leqslant T} |Y_s^n - Y_s|^2 + \mathbf{E}\int_0^T |Z_s^n - Z_s|^2 \,\mathrm{d}s \to 0$$

as $n \to +\infty$.

Let us choose $(X_s^0, Y_s^0, Z_s^0)_{0\leqslant s\leqslant T}$ satisfying the L^p condition. Then, from Pardoux and Peng (1992), we know that for every $n \in \mathbf{N}$, the process $(X_s^n, Y_s^n, Z_s^n)_{0\leqslant s\leqslant T}$ also satisfies the L^p condition.

Let us prove that there exists a constant $\tilde{C}^*_{p,K} > 0$, only depending on p and K, such that for every $T \leqslant \tilde{C}^*_{p,K}$, the sequence $((X^n_s, Y^n_s, Z^n_s)_{0\leqslant s\leqslant T})_{n\in\mathbf{N}}$ satisfies:

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|X^n_s - X^m_s|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y^n_s - Y^m_s|^{2p} + \mathbf{E}\left(\int_0^T |Z^n_s - Z^m_s|^2\,\mathrm{d}s\right)^p \to 0.$$

as $n, m \to +\infty$.

Let us assume that $T \leqslant \tilde{C}_{p,K}$, and fix $n \in \mathbf{N}$. Applying the inequality (A.5.7) to the processes $(X^{n+1}, Y^{n+1}, Z^{n+1})$ and $(X^{n+2}, Y^{n+2}, Z^{n+2})$, we prove that

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|X^{n+2}_s - X^{n+1}_s|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y^{n+2}_s - Y^{n+1}_s|^{2p} + \mathbf{E}\left(\int_0^T |Z^{n+2}_s - Z^{n+1}_s|^2\,\mathrm{d}s\right)^p \\
&\quad\leqslant \tilde{c}_{p,K}\left[\mathbf{E}\left(\int_0^T |f(s, X^{n+1}_s, Y^{n+1}_s, Z^{n+1}_s) - f(s, X^{n+1}_s, Y^n_s, Z^n_s)|\,\mathrm{d}s\right)^{2p}\right. \\
&\qquad\left. + \mathbf{E}\left(\int_0^T |\sigma(s, X^{n+1}_s, Y^{n+1}_s) - \sigma(s, X^{n+1}_s, Y^n_s)|^2\,\mathrm{d}s\right)^p\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|X^{n+2}_s - X^{n+1}_s|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y^{n+2}_s - Y^{n+1}_s|^{2p} + \mathbf{E}\left(\int_0^T |Z^{n+2}_s - Z^{n+1}_s|^2\,\mathrm{d}s\right)^p \\
&\quad\leqslant \tilde{c}_{p,K} K^{2p}(T^p + T^{2p})\left[\mathbf{E}\left(\int_0^T |Z^{n+1}_s - Z^n_s|^2\,\mathrm{d}s\right)^p + \mathbf{E}\sup_{0\leqslant s\leqslant T}|Y^{n+1}_s - Y^n_s|^{2p}\right].
\end{aligned}$$

Therefore, there exists a constant $0 < \tilde{C}^*_{p,K} \leqslant \tilde{C}_{p,K}$, only depending on p and K, such that for $T \leqslant \tilde{C}^*_{p,K}$:

$$\begin{aligned}
&\sum_{n\in\mathbf{N}}\left[\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|X^{n+1}_s - X^n_s|^{2p}\right)^{1/2p} + \left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y^{n+1}_s - Y^n_s|^{2p}\right)^{1/2p}\right. \\
&\quad\left. + \left(\mathbf{E}\left(\int_0^T |Z^{n+1}_s - Z^n_s|^2\,\mathrm{d}s\right)^p\right)^{1/2p}\right] < +\infty.
\end{aligned}$$

This is enough to conclude that for every $T \leqslant \tilde{C}^*_{p,K}$, $(X_s, Y_s, Z_s)_{0\leqslant s\leqslant T}$ satisfies the L^p condition.

Applying the inequality (A.5.7) to the couple $(X_s, Y_s, Z_s)_{0\leqslant s\leqslant T}$ and $(0,0,0)$, we deduce that for every $T \leqslant \tilde{C}^*_{p,K}$:

$$\begin{aligned}
&\mathbf{E}\left(\sup_{0\leqslant s\leqslant T}|X^{0,\xi}_s|^{2p} + \sup_{0\leqslant s\leqslant T}|Y^{0,\xi}_s|^{2p}\right) + \mathbf{E}\left(\int_0^T |Z^{0,\xi}_s|^2\,\mathrm{d}s\right)^p \\
&\quad\leqslant \tilde{c}_{p,K}(\mathbf{E}\,|\xi|^{2p} + \mathbf{E}(\Pi_0)^p). \qquad \text{(A.5.8)}
\end{aligned}$$

Let us show that we can extend the condition $T \leqslant \tilde{C}^*_{p,K}$ to the weaker condition $T \leqslant \tilde{C}^{(2)}_K$. We know that for every $t \in [0,T]$, the random variable $\theta(t,.) : \Omega \times \mathbf{R}^P \to \mathbf{R}^Q$ is $\mathscr{G}^0_t \otimes \mathscr{B}(\mathbf{R}^P)/\mathscr{B}(\mathbf{R}^Q)$ measurable, and satisfies:

$$\text{a.s.,} \quad \forall (x,x') \in (\mathbf{R}^P)^2, \quad |\theta(t,x') - \theta(t,x)|^2 \leqslant \tilde{c}^{(2)}_K |x'-x|^2.$$

Let us consider $(t_i)_{i=0,..,m}$ a subdivision of $[0,T]$ satisfying

$$\forall i \in \{0,..,m-1\}, \quad |t_{i+1} - t_i| \leqslant \tilde{C}^*_{p,\sqrt{\tilde{c}^{(2)}_K}}.$$

Then, from (A.5.8), there exists a constant $\tilde{c}^{(2)}_{p,K}$ only depending on K and p, such that $\forall t_{m-1} \leqslant t \leqslant T$,

$$\mathbf{E}|\theta(t,0)|^{2p} \leqslant \tilde{c}^{(2)}_{p,K}\, \mathbf{E}(|h(0)|^2 + \Pi_{t,T})^p.$$

Therefore, noting that $(X_s, Y_s, Z_s)_{t_{m-2} \leqslant t \leqslant t_{m-1}}$ is the solution of the problem:

$$\begin{cases} \forall t \in [t_{m-2}, t_{m-1}], \\ X_t = X_{t_{m-2}} + \displaystyle\int_{t_{m-2}}^t f(s,X_s,Y_s,Z_s)\,\mathrm{d}s + \int_{t_{m-2}}^t \sigma(s,X_s,Y_s)\,\mathrm{d}B_s, \\ Y_t = \theta(t_{m-1}, X_{t_{m-1}}) + \displaystyle\int_t^{t_{m-1}} g(s,X_s,Y_s,Z_s)\,\mathrm{d}s - \int_t^{t_{m-1}} Z_s\,\mathrm{d}B_s, \\ \mathbf{E}\displaystyle\int_{t_{m-2}}^{t_{m-1}} (|X_t|^2 + |Y_t|^2 + |Z_t|^2)\,\mathrm{d}t < \infty; \end{cases}$$

using once again (A.5.8), and modifying $\tilde{c}^{(2)}_{p,K}$ if necessary, we have $\forall t_{m-2} \leqslant t \leqslant T$,

$$\mathbf{E}|\theta(t,0)|^{2p} \leqslant \tilde{c}^{(2)}_{p,K}\, \mathbf{E}(|h(0)|^2 + \Pi_{t,T})^p.$$

Hence, modifying $\tilde{c}^{(2)}_{p,k}$ if necessary, an induction shows that $\forall 0 \leqslant t \leqslant T$,

$$\mathbf{E}|\theta(t,0)|^{2p} \leqslant \tilde{c}^{(2)}_{p,K}\, \mathbf{E}(|h(0)|^2 + \Pi_{t,T})^p.$$

Now, from (A.5.7) and modifying $\tilde{c}^{(2)}_{p,K}$ if necessary, we have

$$\mathbf{E} \sup_{0 \leqslant s \leqslant t_1} |X_s|^{2p} + \mathbf{E} \sup_{0 \leqslant s \leqslant t_1} |Y_s|^{2p} + \mathbf{E}\left(\int_0^{t_1} |Z_s|^2\,\mathrm{d}s \right)^p$$
$$\leqslant \tilde{c}^{(2)}_{p,K}\, \mathbf{E}(|\xi|^{2p} + (|\theta(t_1,0)|^2 + \Pi_{0,t_1})^p).$$

Using a new induction, we complete the proof of (A.5.1). Using the same scheme, we prove (A.5.2). □

Corollary A.6 (Higher order regularity). *Under Assumption* (A.A1), *for every $1 \leqslant p \leqslant q$, there exist two constants $\tilde{c}^{(1)}_{p,K}$ and $\tilde{c}^{(2)}_{p,K,\Lambda}$, the first one only depending on p and K, and the second one only depending on p, K and Λ, such that for every $T \leqslant \tilde{C}^{(2)}_K$, we have the following*:

For every $(t,x)\in[0,T]\times\mathbf{R}^P$,

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s^{t,x}|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}|^{2p}+\mathbf{E}\left(\int_0^T|Z_s^{t,x}|^2\,\mathrm{d}s\right)^p$$
$$\leqslant\tilde{c}^{(1)}_{p,K}(\mathbf{E}(\Pi_0)^p+|x|^{2p}).\tag{A.6.1}$$

For every $(t,x)\in[0,T]\times\mathbf{R}^P$ *and every* $(t',x')\in[0,T]\times\mathbf{R}^P$,

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|X_s^{t,x}-X_s^{t',x'}|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}-Y_s^{t',x'}|^{2p}+\mathbf{E}\left(\int_0^T|Z_s^{t,x}-Z_s^{t',x'}|^2\,\mathrm{d}s\right)^p$$
$$\leqslant\tilde{c}^{(1)}_{p,K}(|x-x'|^{2p}+\mathbf{E}(\Pi_{t,t'})^p)+\tilde{c}^{(2)}_{p,K,\Lambda}(\mathbf{E}(\Pi_0)^p+|x|^{2p})|t-t'|^p.\tag{A.6.2}$$

Assumption (A.A1′). We say that the functions f, g, h and σ satisfy Assumption (A.A1′) if there exist three constants K, Λ and q such that they satisfy both Assumption (A.A1′) (with respect to the constants K, Λ and q) and the following property:

(A.A1′.1): $\forall t\in[0,T]$, $\forall(x,y,z)\in\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$, $\forall(x',y')\in\mathbf{R}^P\times\mathbf{R}^Q$ and a.s.,

$$|f(t,x',y,z)-f(t,x,y,z)|\leqslant K|x-x'|.$$

That means that the function f is K-lipschitzian with respect to the variables x, y and z.

Let us now consider the following problem:

$$(\mathrm{E}^*)\quad\begin{cases}\forall t\in[0,T],\\ X_t=\xi_t+\displaystyle\int_0^t f(s,X_s,Y_s,Z_s)\,\mathrm{d}s+\int_0^t\sigma(s,X_s,Y_s)\,\mathrm{d}B_s,\\ Y_t=h(X_T)+\displaystyle\int_t^T g(s,X_s,Y_s,Z_s)\,\mathrm{d}s-\int_t^T Z_s\,\mathrm{d}B_s,\\ \mathbf{E}\displaystyle\int_0^T(|X_t|^2+|Y_t|^2+|Z_t|^2)\,\mathrm{d}t<\infty,\end{cases}$$

where $(\xi_s)_{0\leqslant s\leqslant T}$ is a continuous and $\{\mathscr{G}_s^0\}_{0\leqslant s\leqslant T}$ adapted process satisfying:

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|\xi_s|^2<\infty.$$

Actually, problems of type (E^*) are rather useful when using the Malliavin calculus. We state the following theorem:

Theorem A.7. *Assume that* (A.A1′) *is in force. Then, there exists a constant* $C_K^*>0$, *only depending on K, such that for every* $T\leqslant C_K^*$, *for every continuous and* $\{\mathscr{G}_s^0\}_{0\leqslant s\leqslant T}$

adapted process $(\xi_s)_{0\leqslant s\leqslant T}$, *satisfying* $\mathbf{E}\sup_{0\leqslant s\leqslant T}|\xi_s|^2<\infty$, *the problem* $(\mathbf{E}^*)$ *admits a unique solution, denoted* $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$. *It satisfies*:

(i) $(X_t)_{0\leqslant t\leqslant T}$ *and* $(Y_t)_{0\leqslant t\leqslant T}$ *are continuous*
(ii) $\mathbf{E}(\sup_{0\leqslant t\leqslant T}|X_t|^2+\sup_{0\leqslant t\leqslant T}|Y_t|^2)<\infty$.

Moreover, for every $1\leqslant p\leqslant q$, *there exist two constants* $0<C^*_{p,K}\leqslant C^*_K$ *and* $c^*_{p,K}$, *only depending on* p *and* K, *such that for every* $T\leqslant C^*_{p,K}$ *and for every continuous and* $\{\mathscr{G}^0_s\}_{0\leqslant s\leqslant T}$ *adapted process* $(\xi_s)_{0\leqslant s\leqslant T}$, *satisfying* $\mathbf{E}\sup_{0\leqslant s\leqslant T}|\xi_s|^{2p}<\infty$, *the unique solution* $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$ *of the problem* (E^*) *satisfies the following*:

$$\mathbf{E}\left(\sup_{0\leqslant s\leqslant T}|X_s|^{2p}+\sup_{0\leqslant s\leqslant T}|Y_s|^{2p}\right)+\mathbf{E}\left(\int_0^T|Z_s|^2\,\mathrm{d}s\right)^p$$

$$\leqslant c^*_{p,k}\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|\xi_s|^{2p}+\mathbf{E}(\Pi_0)^p\right). \tag{A.7.1}$$

For every vector of function $(\tilde f,\tilde g,\tilde h,\tilde\sigma)$ *satisfying Assumption* $(\mathrm{A.A1}')$ *with respect to the same constants* K, Λ *and* q *as* (f,g,h,σ), *and for every continuous and* $\{\mathscr{G}^0_s\}_{0\leqslant s\leqslant T}$ *adapted process* $(\tilde\xi_s)_{0\leqslant s\leqslant T}$ *satisfying* $\mathbf{E}\sup_{0\leqslant s\leqslant T}|\xi_s|^{2p}<\infty$, *the following estimate holds*:

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde X_s-X_s|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde Y_s-Y_s|^{2p}+\mathbf{E}\left(\int_0^T|\tilde Z_s-Z_s|^2\,\mathrm{d}s\right)^p$$

$$\leqslant c^*_{p,K}\mathbf{E}\Bigg[\sup_{0\leqslant s\leqslant T}|\tilde\xi_s-\xi_s|^{2p}+|\tilde h-h|^{2p}(X_T)$$

$$+\left(\int_0^T(|\tilde f-f|+|\tilde g-g|)(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}$$

$$+\left(\int_0^T|\tilde\sigma-\sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\right)^p\Bigg], \tag{A.7.2}$$

where $(\tilde X_s,\tilde Y_s,\tilde Z_s)_{0\leqslant s\leqslant T}$ *stands for the solution of the problem* (E^*) *associated to the coefficients* $(\tilde f,\tilde g,\tilde h,\tilde\sigma)$ *and to the process* $(\tilde\xi_s)_{0\leqslant s\leqslant T}$.

Proof. We keep the notations given in the statement. Actually, we just have to use the same scheme as in the proof of Theorem A.5, except the estimate of the forward equation. The following method leads to the same kind of inequality as (A.5.3).

We firstly assume that $(X_s,Y_s,Z_s)_{0\leqslant s\leqslant T}$ and $(\tilde X_s,\tilde Y_s,\tilde Z_s)_{0\leqslant s\leqslant T}$ are two solutions of the problem (E^*), as specified in the statement, satisfying the (L_p) and $(\tilde L_p)$ conditions.

Then, using Burkholder–Davis–Gundy inequalities, there exists a constant c_p^*, only depending on p, such that,

$$
\begin{aligned}
\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{X}_s-X_s|^{2p} \leqslant c_p^*\Bigg[&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{\xi}_s-\xi_s|^{2p}\\
&+\mathbf{E}\left(\int_0^T|\tilde{f}(s,\tilde{X}_s,\tilde{Y}_s,\tilde{Z}_s)-f(s,X_s,Y_s,Z_s)|\,\mathrm{d}s\right)^{2p}\\
&+\mathbf{E}\left(\int_0^T|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s)-\sigma(s,X_s,Y_s)|^2\,\mathrm{d}s\right)^{p}\Bigg].
\end{aligned}
$$

Therefore, there exists a constant $c_{p,K}^*$, only depending on p and K, such that,

$$
\begin{aligned}
\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{X}_s-X_s|^{2p} \leqslant c_{p,k}^*\Bigg[&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{\xi}_s-\xi_s|^{2p}\\
&+(T^p+T^{2p})\left(\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{X}_s-X_s|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{Y}_s-Y_s|^{2p}\right)\\
&+T^p\mathbf{E}\left(\int_0^T|\tilde{Z}_s-Z_s|^2\,\mathrm{d}s\right)^{p}\\
&+\mathbf{E}\left(\int_0^T|\tilde{f}-f|(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}\\
&+\mathbf{E}\left(\int_0^T|\tilde{\sigma}-\sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\right)^{p}\Bigg].
\end{aligned}
$$

Hence, modifying $c_{p,K}^*$ if necessary, the same scheme as the one used to show (A.5.2) proves that there exists a constant $C_{p,K}^*>0$ only depending on p and K such that for $T\leqslant C_{p,K}^*$,

$$
\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{X}_s-X_s|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|\tilde{Y}_s-Y_s|^{2p}+\mathbf{E}\left(\int_0^T|\tilde{Z}_s-Z_s|^2\,\mathrm{d}s\right)^{p}\\
&\quad\leqslant c_{p,K}^*\mathbf{E}\Bigg[\sup_{0\leqslant s\leqslant T}|\tilde{\xi}_s-\xi_s|^{2p}+|\tilde{h}-h|^{2p}(X_T)\\
&\qquad+\left(\int_0^T(|\tilde{f}-f|+|\tilde{g}-g|)(s,X_s,Y_s,Z_s)\,\mathrm{d}s\right)^{2p}\\
&\qquad+\left(\int_0^T|\tilde{\sigma}-\sigma|^2(s,X_s,Y_s)\,\mathrm{d}s\right)^{p}\Bigg].
\end{aligned}
$$

Applying the former inequality to the case $p=1$, we deduce from the following iterated scheme that the problem (E*) admits a unique solution if T is small enough:

$$\begin{cases} \forall t \in [0,T], \\ X_t^{n+1} = \xi_t + \int_0^t f(s, X_s^{n+1}, Y_s^n, Z_s^n)\,\mathrm{d}s + \int_0^t \sigma(s, X_s^{n+1}, Y_s^n)\,\mathrm{d}B_s, \\ Y_t^{n+1} = h(X_T^{n+1}) + \int_t^T g(s, X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1})\,\mathrm{d}s + \int_t^T Z_s^{n+1}\,\mathrm{d}B_s. \end{cases}$$

Moreover, following the proof of Theorem A.5, we deduce that (A.7.1) and (A.7.2) hold. □

Appendix B. Regularity of the map θ

As we explain in Appendix A, we want to prove that, under appropriate assumptions and over a small enough time duration T, the map θ given by Corollary 1.5 is a classical solution of the system (E$'$) (Proposition B.6). Using Theorem 6.1 Chapter VII of Ladyzenskaja et al. (1968), we finally deduce from this result that the system (2.1.1) admits a unique bounded classical solution (Corollary B.7).

Actually, we use the same scheme as the one presented by Pardoux and Peng (1992) to build some classical solutions to semi-linear parabolic systems of PDEs by means of stochastic tools. Hence, the two approaches are very close from each other, and this is the reason why we shall not expose ours in details.

Assumption (B.A2). We suppose that f, g, h and σ are deterministic functions satisfying (A1.0) and (A1.3). Moreover, we assume that:

(B.A2.1): f, g, σ and h are continuous on their definition set and twice continuously differentiable with respect to x, y and z.

(B.A2.2): The functions f, g, h and σ are K-lipschitzian with respect to the variables x, y and z, and their derivatives up to order two in the variables x, y and z are K'-lipschitzian with respect to x, y and z (as written in (A1.1)).

Notations. Following Pardoux and Peng (1992), we shall use the notion of derivation on Wiener space, and therefore introduce as explained in Nualart (1995) the space $\mathbf{D}^{1,2}$, as well as the derivated process $(D_r\xi)_{0\leqslant r\leqslant T}$ for a random variable ξ in $\mathbf{D}^{1,2}$.

Moreover, for every $t\in[0,T]$, we denote by $\{\mathscr{F}_s^t\}_{t\leqslant s\leqslant T}$ the filtration $\{\sigma\{B_r - B_t, t \leqslant r \leqslant s\}\}_{t\leqslant s\leqslant T}$ augmented with $\mathscr{N}$.

Theorem B.1. *Suppose that* (B.A2) *is in force. Then, there exists a constant* $0 < \tilde{C}_K^{(3)} \leqslant \tilde{C}_K^{(2)}$ *only depending on K such that for every* $T \leqslant \tilde{C}_K^{(3)}$ *and for every* $(t,x)\in$

$[0,T]\times\mathbf{R}^P$, *the solution* $(X_s^{t,x},Y_s^{t,x},Z_s^{t,x})_{t\leqslant s\leqslant T}$ *of the problem*:

$$\begin{cases}\forall s\in[t,T],\\ X_s^{t,x}=x+\displaystyle\int_t^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,\mathrm{d}r+\int_t^s\sigma(r,X_r^{t,x},Y_r^{t,x})\,\mathrm{d}B_r,\\ Y_s^{t,x}=h(X_T^{t,x})+\displaystyle\int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,\mathrm{d}r-\int_s^T Z_r^{t,x}\,\mathrm{d}B_r,\\ \mathbf{E}\displaystyle\int_t^T(|X_r^{t,x}|^2+|Y_r^{t,x}|^2+|Z_r^{t,x}|^2)\,\mathrm{d}r<\infty\end{cases}$$

satisfies:

(B.1.1) $\forall s\in[t,T],\quad X_s^{t,x}\in(\mathbf{D}^{1,2})^P,\quad Y_s^{t,x}\in(\mathbf{D}^{1,2})^Q;\quad Z^{t,x}\in L^2([t,T],(\mathbf{D}^{1,2})^{Q\times P})$.

In addition, there is a version of $(D_rX_s^{t,x},D_rY_s^{t,x},D_rZ_s^{t,x})_{0\leqslant r\leqslant T,t\leqslant s\leqslant T}$ *satisfying*,

(B.1.2) $\displaystyle\sup_{r\in[0,T]}\left[\mathbf{E}\sup_{t\leqslant s\leqslant T}(|D_rX_s^{t,x}|^2+|D_rY_s^{t,x}|^2)+\mathbf{E}\Big(\int_t^T|D_rZ_s^{t,x}|^2\,\mathrm{d}s\Big)\right]<\infty.$

(B.1.3(i)) $D_rX_s^{t,x}=0,\ D_rY_s^{t,x}=0,\ D_rZ_s^{t,x}=0,\ r\in[0,T]\setminus(t,s]$.
(B.1.3(ii)) $\forall r\in(t,T],\ \forall i\in\{1,..,P\},(D_rX_s^{t,x},D_rY_s^{t,x},D_rZ_s^{t,x})_{r\leqslant s\leqslant T}$ *is the unique solution of the problem* (*which is of type* (E*)) :

$$\begin{cases}\forall s\in[r,T],\\ D_r^iX_s^{t,x}=\sigma_{\cdot,i}(s,X_s^{t,x},Y_s^{t,x})+\displaystyle\int_r^s F^{t,x}(u,D_r^iX_u^{t,x},D_r^iY_u^{t,x},D_r^iZ_u^{t,x})\,\mathrm{d}u\\ \qquad\qquad+\displaystyle\int_r^s\Sigma^{t,x}(u,D_r^iX_u^{t,x},D_r^iY_u^{t,x})\,\mathrm{d}B_u,\\ D_r^iY_s^{t,x}=H^{t,x}(D_r^iX_T^{t,x})+\displaystyle\int_s^T G^{t,x}(u,D_r^iX_u^{t,x},D_r^iY_u^{t,x},D_r^iZ_u^{t,x})\,\mathrm{d}u-\int_s^T D_r^iZ_u^{t,x}\,\mathrm{d}B_u,\\ \mathbf{E}\displaystyle\int_r^T(|D_r^iX_s^{t,x}|^2+|D_r^iY_s^{t,x}|^2+|D_r^iZ_s^{t,x}|^2)\,\mathrm{d}s<\infty,\end{cases}$$

where $F^{t,x}$, $G^{t,x}$, $H^{t,x}$ *and* $\Sigma^{t,x}$ *are defined by the following*: *for every* $(r,u,v,w)\in[t,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$,

$$F^{t,x}(r,u,v,w)=f'_x(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})u+f'_y(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})v$$
$$+f'_z(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})w,$$
$$G^{t,x}(r,u,v,w)=g'_x(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})u+g'_y(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})v$$
$$+g'_z(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})w,$$
$$\Sigma^{t,x}(r,u,v)=\sigma'_x(r,X_r^{t,x},Y_r^{t,x})u+\sigma'_y(r,X_r^{t,x},Y_r^{t,x})v,$$
$$H^{t,x}(u)=h'(X_T^{t,x})u.$$

(B.1.4) *In addition, for any* $1 \leqslant i \leqslant P$, $(D_s^i Y_s^{t,x})_{t\leqslant s\leqslant T}$ *is a version of* $((Z_s^{t,x})^i)_{t\leqslant s\leqslant T}$ (*where* $(Z_s^{t,x})^i$ *denotes the* ith *column of the matrix* $Z_s^{t,x}$).

Sketch of the proof. Let us consider $(t,x)\in[0,T]\times\mathbf{R}^P$. We denote $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{t\leqslant s\leqslant T}$ by $(X_s, Y_s, Z_s)_{t\leqslant s\leqslant T}$.

Then, from Theorem A.5, we know that there exists a constant $\tilde{C}_K^{(3)}>0$, only depending on K, such that for every $T\leqslant \tilde{C}_K^{(3)}$, there exists an approximating sequence $((X_s^n, Y_s^n, Z_s^n)_{t\leqslant s\leqslant T})_{n\in\mathbf{N}}$ of $\{\mathscr{F}_s^t\}_{t\leqslant s\leqslant T}$-progressively measurable processes satisfying:

$$\text{(i)}\quad \begin{cases} \forall s\in[t,T], \\ X_s^{n+1} = x + \displaystyle\int_t^s f(u, X_u^{n+1}, Y_u^n, Z_u^n)\,\mathrm{d}u + \int_t^s \sigma(u, X_u^{n+1}, Y_u^n)\,\mathrm{d}B_u, \\ Y_s^{n+1} = h(X_T^{n+1}) + \displaystyle\int_s^T g(u, X_u^{n+1}, Y_u^{n+1}, Z_u^{n+1})\,\mathrm{d}u - \int_s^T Z_u^{n+1}\,\mathrm{d}B_u. \end{cases}$$

$$\text{(ii)}\quad \sup_{n\in\mathbf{N}}\left[\mathbf{E}\sup_{t\leqslant s\leqslant T}|X_s^n|^4 + \mathbf{E}\sup_{t\leqslant s\leqslant T}|Y_s^n|^4 + \mathbf{E}\left(\int_t^T |Z_s^n|^2\,\mathrm{d}s\right)^2\right] < \infty,$$

As $n,m\to+\infty$,

$$\text{(iii)}\quad \mathbf{E}\sup_{t\leqslant s\leqslant T}|X_s^m - X_s^n|^4 + \mathbf{E}\sup_{t\leqslant s\leqslant T}|Y_s^m - Y_s^n|^4 + \mathbf{E}\left(\int_t^T |Z_s^m - Z_s^n|^2\,\mathrm{d}s\right)^2 \to 0.$$

Now for a fixed $n\in\mathbf{N}$, we consider the following property:

$$\mathscr{H}_n \quad \begin{cases} \forall s\in[t,T],\ X_s^n\in(\mathbf{D}^{1,2})^P,\ Y_s^n\in(\mathbf{D}^{1,2})^Q;\ Z^n\in L^2([t,T],(\mathbf{D}^{1,2})^{Q\times P}). \\ \text{There exists a version of } (D_r X_s^n, D_r Y_s^n, D_r Z_s^n)_{0\leqslant r\leqslant T, t\leqslant s\leqslant T} \text{ satisfying,} \\ \sup_{r\in[0,T]}\left[\mathbf{E}\,\sup_{t\leqslant s\leqslant T}(|D_r X_s^n|^2 + |D_r Y_s^n|^2) + \mathbf{E}\left(\displaystyle\int_t^T |D_r Z_s^n|^2\,\mathrm{d}s\right)\right] < \infty. \end{cases}$$

Following the proof given by Pardoux and Peng (1992), and using their Proposition 2.2, we show that

$$(\mathscr{H}_n) \Rightarrow (\mathscr{H}_{n+1}).$$

Hence, let us choose (X^0, Y^0, Z^0) satisfying $\mathscr{H}_0$. Then, for every $n\in\mathbf{N}$, $\mathscr{H}_n$ holds and

$$\forall r\in[0,T]\setminus(t,s],\quad D_r X_s^n = 0,\quad D_r Y_s^n = 0,\quad D_r Z_s^n = 0.$$

Moreover, for any $r\in(t,T]$, for any $i\in\{i,..,P\}$, $(D_r^i X^{n+1}, D_r^i Y^{n+1}, D_r^i Z^{n+1})$ satisfies the following equation:

$$\text{(iv)}\quad \begin{cases} \forall s\in[r,T], \\ D_r^i X_s^{n+1} = \sigma_{.,i}(s, X_s^{n+1}, Y_s^n) + \displaystyle\int_r^s F_n(u, D_r^i X_u^{n+1}, D_r^i Y_u^n, D_r^i Z_u^n)\,\mathrm{d}u \\ \qquad\qquad + \displaystyle\int_r^s \Sigma_n(u, D_r^i X_u^{n+1}, D_r^i Y_u^n)\,\mathrm{d}B_u, \end{cases}$$

$$D_r^i Y_s^{n+1} = H_n(D_r^i X_T^{n+1}) + \int_s^T G_n(u, D_r^i X_u^{n+1}, D_r^i Y_u^{n+1}, D_r^i Z_u^{n+1})\,\mathrm{d}u$$

$$- \int_s^T D_r^i Z_u^{n+1}\,\mathrm{d}B_u,$$

where for every $n \in \mathbf{N}$, F_n, G_n, H_n and Σ_n are defined by the following: for every $(r,u,v,w) \in [t,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q\times P}$

$$F_n(r,u,v,w) = f'_x(r, X_r^{n+1}, Y_r^n, Z_r^n)u + f'_y(r, X_r^{n+1}, Y_r^n, Z_r^n)v + f'_z(r, X_r^{n+1}, Y_r^n, Z_r^n)w,$$

$$G_n(r,u,v,w) = g'_x(r, X_r^{n+1}, Y_r^{n+1}, Z_r^{n+1})u + g'_y(r, X_r^{n+1}, Y_r^{n+1}, Z_r^{n+1})v$$

$$+ g'_z(r, X_r^{n+1}, Y_r^{n+1}, Z_r^{n+1})w,$$

$$\Sigma_n(r,u,v) = \sigma'_x(r, X_r^{n+1}, Y_r^n)u + \sigma'_y(r, X_r^{n+1}, Y_r^n)v,$$

$$H_n(u) = h'(X_T^{n+1})u.$$

Note that F_n, G_n, Σ_n and H_n are K-lipschitzian with respect to the variables u, v and w. Hence, thanks to Theorem A.7, we deduce, modifying if necessary $\tilde{C}_K^{(3)}$ and the constant $c^*_{2,K}$ given by Theorem A.7, that for every $T \leqslant \tilde{C}_K^{(3)}$, for every $n \in \mathbf{N}$ and for every $r \in (t,T]$:

$$\mathbf{E} \sup_{r\leqslant s\leqslant T} |D_r^i X_s^{n+1}|^4 + \mathbf{E} \sup_{r\leqslant s\leqslant T} |D_r^i Y_s^{n+1}|^4 + \mathbf{E}\left(\int_r^T |D_r^i Z_s^{n+1}|^2\,\mathrm{d}s\right)^2$$

$$\leqslant c^*_{2,K}\left[\Lambda^4 \mathbf{E} \sup_{r\leqslant s\leqslant T} (1 + |X_s^{n+1}| + |Y_s^n|)^4\right.$$

$$\left. + T^2K^4\left(\mathbf{E} \sup_{r\leqslant s\leqslant T} |D_r Y_s^n|^4 + \mathbf{E}\left(\int_r^T |D_r^i Z_s^n|^2\,\mathrm{d}s\right)^2\right)\right].$$

Hence, thanks to (ii), we deduce, modifying $\tilde{C}_K^{(3)}$ if necessary, that for every $T \leqslant \tilde{C}_K^{(3)}$:

$$\text{(v)}\quad \sup_{n\in\mathbf{N}}\left[\sup_{0\leqslant r\leqslant T} \mathbf{E} \sup_{t\leqslant s\leqslant T} |D_r^i X_s^n|^4 + \sup_{0\leqslant r\leqslant T} \mathbf{E} \sup_{t\leqslant s\leqslant T} |D_r^i Y_s^n|^4\right.$$

$$\left. + \sup_{0\leqslant r\leqslant T} \mathbf{E}\left(\int_0^T |D_r^i Z_s^n|^2\,\mathrm{d}s\right)^2\right] < \infty.$$

In the same way, we deduce from Theorem A.7, (B.A2.2) and (iii) that for every $T \leqslant \tilde{C}_K^{(3)}$ (modifying $\tilde{C}_K^{(3)}$ if necessary):

$$\text{(vi)}\quad \sup_{0\leqslant r\leqslant T} \mathbf{E} \sup_{t\leqslant s\leqslant T} |D_r^i X_s^n - D_r^i X_s^m|^2 + \sup_{0\leqslant r\leqslant T} \mathbf{E} \sup_{t\leqslant s\leqslant T} |D_r^i Y_s^n - D_r^i Y_s^m|^2$$

$$+ \sup_{0\leqslant r\leqslant T} \mathbf{E} \int_0^T (|D_r^i Z_s^n - D_r^i Z_s^m|^2)\,\mathrm{d}s \to 0 \quad \text{as } n,m \to +\infty.$$

This shows (B.1.1), (B.1.2) and (B.1.3). Moreover, (B.1.4) is proved as done in Proposition 2.2 of Pardoux and Peng (1992). This completes the proof of Theorem B.1. □

Noting that $F^{t,x}$, $G^{t,x}$, $\Sigma^{t,x}$ and $H^{t,x}$ are K-lipschitzian with respect to u, v and w, we deduce from Theorem A.1 the following proposition.

Proposition B.2. *Under* (B.A2) *for every* $T \leqslant \tilde{C}_K^{(2)}$, *for every* $(t,x) \in \mathbf{R}^P$, *and for every* $i \in \{1,..,P\}$, *the following FBSDE*:

$$\begin{cases} \forall s \in [t,T], \\ \partial_i X_s^{t,x} = e_i + \displaystyle\int_t^s F^{t,x}(u, \partial_i X_u^{t,x}, \partial_i Y_u^{t,x}, \partial_i Z_u^{t,x})\,\mathrm{d}u + \int_t^s \Sigma^{t,x}(u, \partial_i X_u^{t,x}, \partial_i Y_u^{t,x})\,\mathrm{d}B_u, \\ \partial_i Y_s^{t,x} = H^{t,x}(\partial_i X_T^{t,x}) + \displaystyle\int_s^T G^{t,x}(u, \partial_i X_u^{t,x}, \partial_i Y_u^{t,x}, \partial_i Z_u^{t,x})\,\mathrm{d}u - \int_s^T \partial_i Z_u^{t,x}\,\mathrm{d}B_u, \\ \mathbf{E}\displaystyle\int_t^T (|\partial_i X_s^{t,x}|^2 + |\partial_i Y_s^{t,x}|^2 + |\partial_i Z_s^{t,x}|^2)\,\mathrm{d}s < \infty \end{cases}$$

admits a unique solution denoted $(\partial_i X_s^{t,x}, \partial_i Y_s^{t,x}, \partial_i Z_s^{t,x})_{t \leqslant s \leqslant T}$, *where* e_i *stands for the* ith *vector of the canonical basis.*

Proposition B.3. *Assume that* (B.A2) *is in force, and extend the processes* $(X_s^{t,x})_{t\leqslant s\leqslant T}$, $(Y_s^{t,x})_{t\leqslant s\leqslant T}$ *and* $(Z_s^{t,x})_{t\leqslant s\leqslant T}$ *as well as the processes* $(\partial_i X_s^{t,x})_{t\leqslant s\leqslant T, 1\leqslant i\leqslant P}$, $(\partial_i Y_s^{t,x})_{t\leqslant s\leqslant T, 1\leqslant i\leqslant P}$ *and* $(\partial_i Z_s^{t,x})_{t\leqslant s\leqslant T, 1\leqslant i\leqslant P}$ *to the whole interval* $[0,T]$ *as done in Corollary* A.3. *Then, for every* $T \leqslant \tilde{C}_K^{(2)}$, *for every* $p \geqslant 1$, *we have the following properties*:

(B.3.1) *There exists a constant* $\gamma_{p,K}^{(1)}$, *only depending on* p *and* K, *such that* $\forall (t,x,\delta) \in [0,T] \times \mathbf{R}^P \times (\mathbf{R} \setminus \{0\})$, $\forall i \in \{1,..,P\}$,

$$\mathbf{E} \sup_{0\leqslant s\leqslant T} (|\Delta_\delta^i X_s^{t,x}|^{2p} + |\partial_i X_s^{t,x}|^{2p}) + \mathbf{E} \sup_{0\leqslant s\leqslant T} (|\Delta_\delta^i Y_s^{t,x}|^{2p} + |\partial_i Y_s^{t,x}|^{2p})$$
$$+\mathbf{E}\left(\int_0^T (|\partial_i Z_s^{t,x}|^2 + |\Delta_\delta^i Z_s^{t,x}|^2)\,\mathrm{d}s\right)^p \leqslant \gamma_{p,K}^{(1)}.$$

(B.3.2) *There exists a constant* $\gamma_{p,K,K'}^{(2)}$, *only depending on* p, K *and* K' *such that* $\forall (t,x,\delta) \in [0,T] \times \mathbf{R}^P \times (\mathbf{R} \setminus \{0\})$, $\forall i \in \{1,..,P\}$,

$$\mathbf{E} \sup_{0\leqslant s\leqslant T} |\Delta_\delta^i X_s^{t,x} - \partial_i X_s^{t,x}|^{2p} + \mathbf{E} \sup_{0\leqslant s\leqslant T} |\Delta_\delta^i Y_s^{t,x} - \partial_i Y_s^{t,x}|^{2p}$$
$$+\mathbf{E}\left(\int_0^T |\Delta_\delta^i Z_s^{t,x} - \partial_i Z_s^{t,x}|^2\,\mathrm{d}s\right)^p \leqslant \gamma_{p,K,K'}^{(2)} |\delta|^{2p},$$

where, for a function ℓ *of* $x \in \mathbf{R}^P$, *for* $\delta \in \mathbf{R} \setminus \{0\}$ *and for* $i \in \{1,..,P\}$,

$$\Delta_\delta^i \ell(x) = \frac{1}{\delta}\,[\ell(x + \delta e_i) - \ell(x)].$$

(B.3.3) *There exist two constants* $\gamma^{(3)}_{p,K,K'}$ *and* $\gamma^{(4)}_{p,K,K',\Lambda'}$, *the first one only depending on* p, K *and* K' *and the second one only depending on* p, K, K' and Λ *such that* $\forall (t,x) \in [0,T] \times \mathbf{R}^P$, $\forall (t',x') \in [0,T] \times \mathbf{R}^P$, $\forall i \in \{1,..,P\}$,

$$\mathbf{E} \sup_{0 \leqslant s \leqslant T} |\partial_i X_s^{t,x} - \partial_i X_s^{t',x'}|^{2p} + \mathbf{E} \sup_{0 \leqslant s \leqslant T} |\partial_i Y_s^{t,x} - \partial_i Y_s^{t',x'}|^{2p}$$

$$+ \mathbf{E}\left(\int_0^T |\partial_i Z_s^{t,x} - \partial_i Z_s^{t',x'}|^2 \,\mathrm{d}s \right)^p$$

$$\leqslant \gamma^{(3)}_{p,K,K'} |x - x'|^{2p} + \gamma^{(4)}_{p,K,K',\Lambda} (1 + |x|^{2p}) |t - t'|^p.$$

Sketch of the proof. Let us consider $p \geqslant 1$. Following Pardoux and Peng (1992), we notice that for every $(t,x,\delta) \in [0,T] \times \mathbf{R}^P \times \mathbf{R} \setminus \{0\}$ and for every $i \in \{1,..,P\}$, the process $(\Delta^i_\delta X_s^{t,x}, \Delta^i_\delta Y_s^{t,x}, \Delta^i_\delta Z_s^{t,x})_{0 \leqslant s \leqslant T}$ satisfies the FBSDE:

$$\begin{cases} \forall s \in [0,T], \\ \Delta^i_\delta X_s^{t,x} = e_i + \displaystyle\int_0^s \mathbf{1}_{[t,T]}(r) \left(\int_0^1 [f'_x(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta X_r^{t,x} + f'_y(\zeta^{t,x,\lambda}_{r,\lambda}) \Delta^i_\delta Y_r^{t,x} \right. \\ \qquad\qquad \left. + f'_z(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta Z_r^{t,x}] \,\mathrm{d}\lambda \right) \mathrm{d}r \\ \qquad\qquad + \displaystyle\int_0^s \mathbf{1}_{[t,T]}(r) \left(\int_0^1 [\sigma'_x(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta X_r^{t,x} + \sigma'_y(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta Y_r^{t,x}] \,\mathrm{d}\lambda \right) \mathrm{d}B_r, \\ \Delta^i_\delta Y_s^{t,x} = \displaystyle\int_0^1 [h'(X_T^{t,x} + \lambda\delta \Delta_\delta X_T^{t,x}) \Delta_\delta X_T^{t,x}] \,\mathrm{d}\lambda \\ \qquad\qquad + \displaystyle\int_s^T \mathbf{1}_{[t,T]}(r) \left(\int_0^1 [g'_x(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta X_r^{t,x} \right. \\ \qquad\qquad \left. + g'_y(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta Y_r^{t,x} + g'_z(\zeta^{t,x,\delta}_{r,\lambda}) \Delta^i_\delta Z_r^{t,x}] \,\mathrm{d}\lambda \right) \mathrm{d}r - \displaystyle\int_s^T \Delta^i_\delta Z_r^{t,x} \,\mathrm{d}B_r, \end{cases}$$

where

$$\zeta^{t,x,\delta}_{r,\lambda} = (r, X_r^{t,x} + \lambda\delta \Delta^i_\delta X_r^{t,x}, Y_r^{t,x} + \lambda\delta \Delta^i_\delta Y_r^{t,x}, Z_r^{t,x} + \lambda\delta \Delta^i_\delta Z_r^{t,x}).$$

Then, noting that the functions:

$$F^{\delta,t,x} : [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q \times P} \to \mathbf{R}^P$$

$$(r,u,v,w) \mapsto \mathbf{1}_{[t,T]}(r) \left(\int_0^1 [f'_x(\zeta^{t,x,\delta}_{r,\lambda}) u + f'_y(\zeta^{t,x,\delta}_{r,\lambda}) v + f'_z(\zeta^{t,x,\delta}_{r,\lambda}) w] \,\mathrm{d}\lambda \right),$$

$$G^{\delta,t,x} : [0,T] \times \mathbf{R}^P \times \mathbf{R}^Q \times \mathbf{R}^{Q \times P} \to \mathbf{R}^Q$$

$$(r,u,v,w)\mapsto \mathbf{1}_{[t,T]}(r)\left(\int_0^1 [g'_x(\zeta^{t,x,\delta}_{r,\lambda})u+g'_y(\zeta^{t,x,\delta}_{r,\lambda})v+g'_z(\zeta^{t,x,\delta}_{r,\lambda})w]\,\mathrm{d}\lambda\right),$$

$$\Sigma^{\delta,t,x}:[0,T]\times\mathbf{R}^P\times\mathbf{R}^Q\to\mathbf{R}^P$$

$$(r,u,v)\mapsto \mathbf{1}_{[t,T]}(r)\left(\int_0^1 [\sigma'_x(\zeta^{t,x,\delta}_{r,\lambda})u+\sigma'_y(\zeta^{t,x,\delta}_{r,\lambda})v]\,\mathrm{d}\lambda\right),$$

and

$$H^{\delta,t,x}:\mathbf{R}^P\to\mathbf{R}^Q$$

$$u\mapsto \int_0^1 [h'(X^{t,x}_T+\lambda\delta\Delta^i_\delta X^{t,x}_T)u]\,\mathrm{d}\lambda,$$

as well as the functions $(F^{t,x},G^{t,x},\Sigma^{t,x},H^{t,x})$ satisfy Assumption (A.A1) with respect to the constant K (K plays the roles of K and Λ), we easily deduce, thanks to Theorem A.5 (B.3.1).

Let us prove (B.3.2). Using once again Theorem A.5, we prove that there exists a constant $\gamma'_{p,K}$, only depending on p and K, such that for every $(t,x)\in[0,T]\times\mathbf{R}^P$,

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta^i_\delta X^{t,x}_s-\partial_i X^{t,x}_s|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta^i_\delta Y^{t,x}_s-\partial_i Y^{t,x}_s|^{2p}\\
&\quad+\mathbf{E}\left(\int_0^T|\Delta^i_\delta Z^{t,x}_s-\partial_i Z^{t,x}_s|^2\,\mathrm{d}s\right)^p\\
&\quad\leqslant \gamma'_{p,K}\Bigg[\mathbf{E}|(H^{\delta,t,x}-H^{t,x})(\partial_i X^{t,x}_T)|^{2p}\\
&\qquad+\mathbf{E}\left(\int_t^T|\Sigma^{\delta,t,x}-\Sigma^{t,x}|^2(r,\partial_i X^{t,x}_r,\partial_i Y^{t,x}_r)\,\mathrm{d}r\right)^p\\
&\qquad+\mathbf{E}\left(\int_t^T(|F^{\delta,t,x}-F^{t,x}|+|G^{\delta,t,x}-G^{t,x}|)(r,\partial_i X^{t,x}_r,\partial_i Y^{t,x}_r,\partial_i Z^{t,x}_r)\,\mathrm{d}r\right)^{2p}\Bigg].
\end{aligned}$$

Hence, thanks to Assumption (B.A2), there exits a constant $\gamma''_{p,K,K'}$, only depending on K, K' and p such that:

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta^i_\delta X^{t,x}_s-\partial_i X^{t,x}_s|^{2p}+\mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta^i_\delta Y^{t,x}_s-\partial_i Y^{t,x}_s|^{2p}\\
&\quad+\mathbf{E}\left(\int_0^T|\Delta^i_\delta Z^{t,x}_s-\partial_i Z^{t,x}_s|^2\,\mathrm{d}s\right)^p\\
&\quad\leqslant \gamma''_{p,K,K'}|\delta|^{2p}\Bigg[\mathbf{E}(|\Delta^i_\delta X^{t,x}_T|\,|\partial_i X^{t,x}_T|)^{2p}\\
&\qquad+\mathbf{E}\left(\int_0^T(|\Delta^i_\delta X^{t,x}_s|^2+|\Delta^i_\delta Y^{t,x}_s|^2)(|\partial_i X^{t,x}_s|^2+|\partial_i Y^{t,x}_s|^2)\,\mathrm{d}s\right)^p
\end{aligned}$$

$$+\mathbf{E}\left(\int_0^T (|\Delta_\delta^i X_s^{t,x}| + |\Delta_\delta^i Y_s^{t,x}| + |\Delta_\delta^i Z_s^{t,x}|)(|\partial_i X_s^{t,x}| + |\partial_i Y_s^{t,x}| + |\partial_i Z_s^{t,x}|)\,\mathrm{d}s\right)^{2p}\Bigg].$$

Therefore, using (B.3.1) (with respect to $2p$ instead of p), and modifying $\gamma''_{p,K,K'}$ if necessary, it is not quite hard to deduce that for every $(t,x)\in[0,T]\times\mathbf{R}^P$,

$$\mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta_\delta^i X_s^{t,x} - \partial_i X_s^{t,x}|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|\Delta_\delta^i Y_s^{t,x} - \partial_i Y_s^{t,x}|^{2p} + \mathbf{E}\left(\int_0^T |\Delta_\delta^i Z_s^{t,x} - \partial_i Z_s^{t,x}|^2\,\mathrm{d}s\right)^p \leqslant \gamma''_{p,K,K'}|\delta|^{2p}.$$

Moreover, form Theorem A.5, there exists a constant $\gamma'''_{p,K,K'}$, only depending on p, K and K' such that for every $(t,x)\in[0,T]\times\mathbf{R}^P$ and for every $(t',x')\in[0,T]\times\mathbf{R}^P$, $0\leqslant t\leqslant t'\leqslant T$,

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\partial_i X_s^{t',x'} - \partial_i X_s^{t,x}|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|\partial_i Y_s^{t',x'} - \partial_i Y_s^{t,x}|^{2p} + \mathbf{E}\left(\int_0^T |\partial_i Z_s^{t',x'} - \partial_i Z_s^{t,x}|^2\,\mathrm{d}s\right)^p\\
&\leqslant \gamma'''_{p,K,K'}\Bigg[(t'-t)^p + \mathbf{E}(|X_T^{t,x} - X_T^{t',x'}|\,|\partial_i X_T^{t,x}|)^{2p}\\
&\quad + \mathbf{E}\left(\int_{t'}^T (|X_s^{t,x} - X_s^{t',x'}| + |Y_s^{t,x} - Y_s^{t',x'}| + |Z_s^{t,x} - Z_s^{t',x'}|)\times(|\partial_i X_s^{t,x}| + |\partial_i Y_s^{t,x}| + |\partial_i Z_s^{t,x}|)\,\mathrm{d}s\right)^{2p}\\
&\quad + \mathbf{E}\left(\int_{t'}^T (|X_s^{t,x} - X_s^{t',x'}|^2 + |Y_s^{t,x} - Y_s^{t',x'}|^2)(|\partial_i X_s^{t,x}|^2 + |\partial_i Y_s^{t,x}|^2)\,\mathrm{d}s\right)^p\Bigg].
\end{aligned}$$

Therefore, from Corollary A.6, there exist two constants $\gamma^{(3)}_{p,K,K'}$ and $\gamma^{(4)}_{p,K,K',\Lambda}$, the first one only depending on p, K and K', and the second one only depending on p, K, K' and Λ such that for every $(t,x)\in[0,T]\times\mathbf{R}^P$ and for every $(t',x')\in[0,T]\times\mathbf{R}^P$, $0\leqslant t\leqslant t'\leqslant T$,

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant s\leqslant T}|\partial_i X_s^{t',x'} - \partial_i X_s^{t,x}|^{2p} + \mathbf{E}\sup_{0\leqslant s\leqslant T}|\partial_i Y_s^{t',x'} - \partial_i Y_s^{t,x}|^{2p} + \mathbf{E}\left(\int_0^T |\partial_i Z_s^{t',x'} - \partial_i Z_s^{t,x}|^2\,\mathrm{d}s\right)^p\\
&\leqslant \gamma^{(3)}_{p,K,K'}|x-x'|^{2p} + \gamma^{(4)}_{p,K,K',\Lambda}(1+|x|^{2p})(t'-t)^p. \qquad \square
\end{aligned}$$

We deduce the following corollary:

Corollary B.4. *Under Assumption* (B.A2), *for every* $T \leqslant \tilde{C}_K^{(2)}$, *the map* θ *is twice differentiable with respect to* x, *and the functions* θ, $(\partial\theta/\partial x_i)_i$ *and* $(\partial^2\theta/\partial x_i\,\partial x_j)_{i,j}$ *are continuous on* $[0,T]\times\mathbf{R}^P$.

Moreover, there exists a constant $\gamma_{K,K'}^{(5)}$, *only depending on* K *and* K', *such that* $(\partial\theta/\partial x_i)_i$ *and* $(\partial^2\theta/\partial x_i\,\partial x_j)_{i,j}$ *are* $\gamma_{K,K'}^{(5)}$*-lipschitzian with respect to* x (*from property* (A.3.2) *of Corollary* A.3, *we recall that* θ *is* $\tilde{c}_K^{(2)}$*-lipschitzian*).

Sketch of the proof. Thanks to property (B.3.2) of Proposition B.3, θ is differentiable with respect to x and its partial derivatives are given by:

$$\forall j\in\{1,..,P\},\ \forall (t,x)\in[0,T]\times\mathbf{R}^P,\ \frac{\partial\theta}{\partial x_j}(t,x)=\partial_j Y_t^{t,x}.$$

Moreover, from property (B.3.3), for every $j\in\{1,..,P\}$, the function $\partial\theta/\partial x_j$ is continuous and $\gamma_{K,K'}^{(3)}$-lispchitzian with respect to the variable x. Actually, thanks to Assumption (B.A2), the same scheme can be applied to prove that the functions $(\partial\theta/\partial x_j)_j$ are continuously differentiable with respect to the variable x. Indeed, for every $T\leqslant\tilde{C}_K^{(2)}$, for every $(t,x)\in[0,T]\times\mathbf{R}^P$ and for every $(i,j)\in\{1,..,P\}^2$, the following FBSDE admits a unique solution:

$$\begin{cases} \forall s\in[t,T],\\ \partial_{i,j}^2 X_s^{t,x} = \displaystyle\int_t^s F_{2,i,j}^{t,x}(u,\partial_{i,j}^2 X_u^{t,x},\partial_{i,j}^2 Y_u^{t,x},\partial_{i,j}^2 Z_u^{t,x})\,\mathrm{d}u\\ \qquad\qquad + \displaystyle\int_t^s \Sigma_{2,i,j}^{t,x}(u,\partial_{i,j}^2 X_u^{t,x},\partial_{i,j}^2 Y_u^{t,x})\,\mathrm{d}B_u,\\ \partial_{i,j}^2 Y_s^{t,x} = H_{2,i,j}^{t,x}(\partial_{i,j}^2 X_T^{t,x}) + \displaystyle\int_s^T G_{2,i,j}^{t,x}(u,\partial_{i,j}^2 X_u^{t,x},\partial_{i,j}^2 Y_u^{t,x},\partial_{i,j}^2 Z_u^{t,x})\,\mathrm{d}u\\ \qquad\qquad - \displaystyle\int_s^T \partial_{i,j}^2 Z_u^{t,x}\,\mathrm{d}B_u,\\ \mathbf{E}\displaystyle\int_t^T (|\partial_{i,j}^2 X_s^{t,x}|^2+|\partial_{i,j}^2 Y_s^{t,x}|^2+|\partial_{i,j}^2 Z_s^{t,x}|^2)\,\mathrm{d}s<\infty, \end{cases}$$

where, for every $(r,u_1,u_2,u_3)\in[t,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$:

$$\begin{aligned} F_{2,i,j}^{t,x}(r,u_1,u_2,u_3) &= \sum_{p=1}^{3} f'_{w_p}(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})u_p\\ &\quad + \sum_{p,q=1}^{3} (f''_{w_p,w_q}(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\partial_i W_r^{p,t,x})\partial_j W_r^{q,t,x}, \end{aligned}$$

$$G^{t,x}_{2,i,j}(r,u_1,u_2,u_3) = \sum_{p=1}^{3} g'_{w_p}(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)u_p + \sum_{p,q=1}^{3}(g''_{w_p,w_q}(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)\partial_i W^{p,t,x}_r)\partial_j W^{q,t,x}_r,$$

$$\Sigma^{t,x}_{2,i,j}(r,u_1,u_2) = \sum_{p=1}^{2} \sigma'_{w_p}(r,X^{t,x}_r,Y^{t,x}_r)u_p + \sum_{p,q=1}^{2}(\sigma''_{w_p,w_q}(r,X^{t,x}_r,Y^{t,x}_r)\partial_i W^{p,t,x}_r)\partial_j W^{q,t,x}_r,$$

$$H^{t,x}_{2,i,j}(u_1) = h'(X^{t,x}_T)u_1 + (h''(X^{t,x}_T)\partial_i X^{t,x}_T)\partial_j X^{t,x}_T,$$

where, for the sake of simplicity, we have denoted by $(W^{1,t,x},W^{2,t,x},W^{3,t,x})$ the process $(X^{t,x},Y^{t,x},Z^{t,x})$ and by (w_1,w_2,w_3) the triple (x,y,z). Moreover, let us note that for every $\delta\in\mathbf{R}\setminus\{0\}$, the triple $(\Delta^i_\delta\partial_j X^{t,x},\Delta^i_\delta\partial_j Y^{t,x},\Delta^i_\delta\partial_j Z^{t,x})$ satisfies the FBSDE:

$$\forall s\in[t,T],$$

$$\Delta^i_\delta\partial_j X^{t,x}_s = \int_t^s F^{t,x,\delta}_{2,i,j}(u,\Delta^i_\delta\partial_j X^{t,x}_u,\Delta^i_\delta\partial_j Y^{t,x}_u,\Delta^i_\delta\partial_j Z^{t,x}_u)\,\mathrm{d}u + \int_t^s \Sigma^{t,x,\delta}_{2,i,j}(u,\Delta^i_\delta\partial_j X^{t,x}_u,\Delta^i_\delta\partial_j Y^{t,x}_u)\,\mathrm{d}B_u,$$

$$\Delta^i_\delta\partial_j Y^{t,x}_s = H^{t,x,\delta}_{2,i,j}(\Delta^i_\delta\partial_j X^{t,x}_T) + \int_s^T G^{t,x,\delta}_{2,i,j}(u,\Delta^i_\delta\partial_j X^{t,x}_u,\Delta^i_\delta\partial_j Y^{t,x}_u,\Delta^i_\delta\partial_j Z^{t,x}_u)\,\mathrm{d}u - \int_s^T \Delta^i_\delta\partial_j Z^{t,x}_u\,\mathrm{d}B_u,$$

$$\mathbf{E}\int_t^T(|\Delta^i_\delta\partial_j X^{t,x}_s|^2 + |\Delta^i_\delta\partial_j Y^{t,x}_s|^2 + |\Delta^i_\delta\partial_j Z^{t,x}_s|^2)\,\mathrm{d}s<\infty,$$

where, for every $(r,u_1,u_2,u_3)\in[t,T]\times\mathbf{R}^P\times\mathbf{R}^Q\times\mathbf{R}^{Q\times P}$,

$$F^{t,x,\delta}_{2,i,j}(r,u_1,u_2,u_3) = \sum_{p=1}^{3} f'_{w_p}(r,X^{t,x+\delta e_i}_r,Y^{t,x+\delta e_i}_r,Z^{t,x+\delta e_i}_r)u_p + \sum_{p,\,q=1}^{3}\int_0^1(f''_{w_p,w_q}(\zeta^{t,x,\delta}_{r,\lambda})\Delta^i_\delta W^{p,t,x}_r)\partial_j W^{q,t,x}_r\,\mathrm{d}\lambda,$$

$$G_{2,i,j}^{t,x,\delta}(r,u_1,u_2,u_3) = \sum_{p=1}^{3} g'_{w_p}(r,X_r^{t,x+\delta e_i}, Y_r^{t,x+\delta e_i}, Z_r^{t,x+\delta e_i})u_p$$

$$+ \sum_{p,q=1}^{3} \int_0^1 (g''_{w_p,w_q}(\zeta_{r,\lambda}^{t,x,\delta})\Delta_\delta^i W_r^{p,t,x})\partial_j W_r^{q,t,x}\,\mathrm{d}\lambda,$$

$$\Sigma_{2,i,j}^{t,x,\delta}(r,u_1,u_2) = \sum_{p=1}^{2} \sigma'_{w_p}(r,X_r^{t,x+\delta e_i}, Y_r^{t,x+\delta e_i})u_p$$

$$+ \sum_{p,q=1}^{2} \int_0^1 (\sigma''_{w_p,w_q}(\zeta_{r,\lambda}^{t,x,\delta})\Delta_\delta^i W_r^{p,t,x})\partial_j W_r^{q,t,x}\,\mathrm{d}\lambda,$$

$$H_{2,i,j}^{t,x,\delta}(u_1) = h'(X_T^{t,x+\delta e_i})u_1 + \int_0^1 (h''(X_T^{t,x} + \lambda\delta\Delta_\delta^i X_T^{t,x})\Delta_\delta^i X_T^{t,x})\partial_j X_t^{t,x}\,\mathrm{d}\lambda.$$

Following the proof of Proposition 5.3, we deduce that the function $\partial\theta/\partial x_j$ is differentiable with respect to x, and its partial derivatives, given by

$$\forall j \in \{1,..,P\},\ \forall (t,x) \in [0,T] \times \mathbf{R}^P, \quad \frac{\partial^2 \theta}{\partial x_i\,\partial x_j}(t,x) = \partial_{i,j}^2 Y_t^{t,x}$$

are continuous and $\gamma_{K,K'}^{(5)}$-lipschitzian with respect to x, where $\gamma_{K,K'}^{(5)}$ only depends on K and K'. $\square$

From property (B.1.4) of Theorem B.1 and Corollary B.4, we deduce:

Corollary B.5. *For every $T \leqslant \tilde{C}_K^{(3)}$ and for every $(t,x) \in [0,T] \times \mathbf{R}^P, (Z_s^{t,x})_{t\leqslant s\leqslant T}$ has a continuous version, with which we identify it, and which is given by*

$$\forall s \in [t,T], \quad Z_s^{t,x} = \nabla_x \theta(s,X_s^{t,x})\sigma(s,X_s^{t,x},Y_s^{t,x}) = \nabla_x\theta(s,X_s^{t,x})\tilde{\sigma}(s,X_s^{t,x}),$$

where

$$\forall (t,x) \in [0,T] \times \mathbf{R}^P, \quad \tilde{\sigma}(t,x) = \sigma(t,x,\theta(t,x)).$$

Let us conclude with the next proposition.

Proposition B.6. *Under Assumption* (B.A2), *for every $T \leqslant \tilde{C}_K^{(2)}$, the map $(t,x) \mapsto Y_t^{t,x}$ belongs to $\mathbf{C}^{1,2}([0,T] \times \mathbf{R}^P, \mathbf{R}^Q)$ and satisfies the system of PDEs* ($\mathbf{E}'$).

Proof. Let us consider $T \leqslant \tilde{C}_K^{(3)}$, $(t,x) \in [0,T[\times \mathbf{R}^P$, and $\delta > 0$, such that $t+\delta \leqslant T$. Then, for every $k \in \{1,..,Q\}$,

$$\theta_k(t+\delta,x) - \theta_k(t,x) = \theta_k(t+\delta,x) - \theta_k(t+\delta,X_{t+\delta}^{t,x}) + \theta_k(t+\delta,X_{t+\delta}^{t,x}) - \theta_k(t,x)$$

$$= -\sum_{i=1}^{P} \int_t^{t+\delta} \frac{\partial \theta_k}{\partial x_i}(t+\delta,X_s^{t,x})f_i(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,\mathrm{d}s$$

$$-\int_t^{t+\delta}\sum_{i=1}^{P}\frac{\partial\theta_k}{\partial x_i}(t+\delta,X_s^{t,x})(\sigma(s,X_s^{t,x},Y_s^{t,x})\,\mathrm{d}B_s)_i$$

$$-\int_t^{t+\delta}\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2\theta_k}{\partial x_i\,\partial x_j}(t+\delta,X_s^{t,x})a_{i,j}(s,X_s^{t,x},Y_s^{t,x})\,\mathrm{d}s$$

$$-\int_t^{t+\delta}g_k(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,\mathrm{d}s+\int_t^{t+\delta}(Z_s^{t,x}\,\mathrm{d}B_s)_k,$$

where we have used Itô's formula for the function $\theta(t+\delta,.)$. Hence, from Corollary B.5, for every $t=t_0<t_1<\cdots<t_n=T$,

$$\begin{aligned}h_k(x)-\theta_k(t,x)=&-\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\Bigg[\sum_{i=1}^{P}\frac{\partial\theta_k}{\partial x_i}(t_{\ell+1},X_s^{t_\ell,x})f_i(s,X_s^{t_\ell,x},Y_s^{t_\ell,x},\nabla_x\theta(s,X_s^{t_\ell,x})\tilde{\sigma}(s,X_s^{t_\ell,x}))\\&+\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2\theta_k}{\partial x_i\,\partial x_j}(t_{\ell+1},X_s^{t_l,x})a_{i,j}(s,X_s^{t_\ell,x},Y_s^{t_\ell,x})\\&+g_k(s,X_s^{t_\ell,x},Y_s^{t_\ell,x},\nabla_x\theta(s,X_s^{t_\ell,x})\tilde{\sigma}(s,X_s^{t_\ell,x}))\Bigg]\,\mathrm{d}s\\&+\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}((\nabla_x\theta(s,X_s^{t_\ell,x})-\nabla_x\theta(t_{\ell+1},X_s^{t_\ell,x}))\tilde{\sigma}(s,X_s^{t_\ell,x})\,\mathrm{d}B_s)_k.\end{aligned}$$

From Corollary A.6, we prove that the processes $(X_s^{t,x})_{0\leqslant s\leqslant T,0\leqslant t\leqslant T,x\in\mathbf{R}^P}$ and $(Y_s^{t,x})_{0\leqslant s\leqslant T,0\leqslant t\leqslant T,x\in\mathbf{R}^P}$ are continuous on $[0,T]\times[0,T]\times\mathbf{R}^P$.

Therefore, using Corollary B.4, and taking a sequence of subdivisions $((t_0^n,..,t_n^n))_{n\in\mathbf{N}}$ such that $\lim_{n\to\infty}\sup_{0\leqslant i\leqslant n-1}(t_{i+1}^n-t_i^n)=0$, we obtain

$$\begin{aligned}\theta_k(t,x)=h_k(x)+\int_t^T&\Bigg[\sum_{i=1}^{P}\frac{\partial\theta_k}{\partial x_i}(s,x)f_i(s,x,\theta(s,x),\nabla_x\theta(s,x)\sigma(s,x,\theta(s,x)))\\&+\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2\theta_k}{\partial x_i\,\partial x_j}(s,x)a_{i,j}(s,x,\theta(s,x))\\&+g_k(s,x,\theta(s,x),\nabla_x\theta(s,x)\sigma(s,x,\theta(s,x)))\Bigg]\,\mathrm{d}s.\end{aligned}$$

Considering a subdivision $(\tau_i)_{i=1,..,m}$ of $[0,T]$ such that

$$\forall i\in\{1,..,m-1\},\quad|\tau_{i+1}-\tau_i|\leqslant\tilde{C}^{(3)}_{\tilde{C}_K^{(2)}}$$

we obtain, using an induction, the result for $T\leqslant\tilde{C}_K^{(2)}$. □

Corollary B.7. *Under Assumption* (A2) *and Assumption* (B.A2), *for any* $T>0$, *the system* (E′) *has a unique bounded classical solution.*

Proof. Let us introduce the constants $\tilde{C}$ and $\tilde{\Gamma}$, defined in (2.1.3) and (2.1.4), and let us set:

$$\tilde{K} = \max(k, K, \tilde{\Gamma}), \quad \gamma = \tilde{C}^{(2)}_{\tilde{K}}.$$

Therefore, from Proposition B.6, we build $u : [T-\gamma, T] \times \mathbf{R}^P \to \mathbf{R}^Q$ a classical solution of the problem:

$$\begin{cases} \forall (t,x) \in [T-\gamma, T] \times \mathbf{R}^P, \ \forall \ell \in \{1, .., Q\}, \\ \dfrac{\partial \theta_\ell}{\partial t}(t,x) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{P} a_{i,j}(t,x,\theta(t,x)) \dfrac{\partial^2 \theta_\ell}{\partial x_i \, \partial x_j}(t,x) \\ + \displaystyle\sum_{i=1}^{P} f_i(t,x,\theta(t,x), \nabla_x \theta(t,x) \sigma(t,x,\theta(t,x))) \dfrac{\partial \theta_\ell}{\partial x_i}(t,x) \\ + g_\ell(t,x,\theta(t,x), \nabla_x \theta(t,x) \sigma(t,x,\theta(t,x))) = 0, \\ \forall x \in \mathbf{R}^P, \ \theta(T,x) = h(x). \end{cases}$$

Following the proof of Lemma 2.1, we are able to prove that u is bounded by $\tilde{C}$ (note that u is a priori not bounded, but due to the stochastic representation of u, we can prove that it is). Thanks to Theorem 6.1 Chapter VII of Ladyzenskaja et al. (1968), we deduce that $|\nabla_x u|$ is bounded by $\tilde{\Gamma}$. Moreover, as a consequence of Corollary B.4, the derivatives of order one and two of u with respect to x are Lipschitzian with respect to x. In particular, the function $u(t_{N-1}, .)$ is $\tilde{\Gamma}$ lipschitzian and its derivatives of order one and two are also lipschitzian.

Hence, considering the problem:

$$\begin{cases} \forall (t,x) \in [T-2\gamma, T-\gamma] \times \mathbf{R}^P, \ \forall \ell \in \{1, .., Q\}, \\ \dfrac{\partial \theta_\ell}{\partial t}(t,x) + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{P} a_{i,j}(t,x,\theta(t,x)) \dfrac{\partial^2 \theta_\ell}{\partial x_i \, \partial x_j}(t,x) \\ + \displaystyle\sum_{i=1}^{P} f_i(t,x,\theta(t,x), \nabla_x \theta(t,x) \sigma(t,x,\theta(t,x))) \dfrac{\partial \theta_\ell}{\partial x_i}(t,x) \\ + g_\ell(t,x,\theta(t,x), \nabla_x \theta(t,x) \sigma(t,x,\theta(t,x))) = 0, \\ \forall x \in \mathbf{R}^P, \ \theta(T-\gamma,x) = u(T-\gamma,x), \end{cases}$$

we are able to extend from Proposition B.6 the function u to the set $[T-2\gamma, T] \times \mathbf{R}^P$. An induction completes the proof of existence.

Let us prove uniqueness. Let v be a bounded solution of the system (E′). Thanks again to Theorem 6.1 Chapter VII of Ladyzenskaja et al. (1968), we know that $\nabla_x v$ is bounded on $[0,T] \times \mathbf{R}^P$. Then, thanks to property (A2.3) and to Corollary 3.23 Chapter V of Karatzas and Shreve (1988) (Strong uniqueness and weak existence of SDE), we can associate to v a family of processes $(X^{v,t,x})_{(t,x)\in[0,T]\times\mathbf{R}^P}$, solutions of the SDEs:

$$\forall s\in[t,T],$$

$$X_s^{v,t,x}=x+\int_t^s f(r,X_r^{v,t,x},v(r,X_r^{v,t,x}),(\nabla_x v\sigma)(r,X_r^{v,t,x},v(r,X_r^{v,t,x}))\,\mathrm{d}r$$
$$+\int_t^s \sigma(r,X_r^{v,t,x},v(r,X_r^{v,t,x}))\,\mathrm{d}B_r.$$

Hence, denoting for every $(t,x)\in[0,T]\times\mathbf{R}^P$, for every $s\in[t,T]$, $Y_s^{v,t,x}=v(s,X_s^{v,t,x})$ and $Z_s^{v,t,x}=\nabla_x v(s,X_s^{v,t,x})\sigma(s,X_s^{v,t,x},Y_s^{v,t,x})$, we deduce from the Itô formula that for every $s\in[t,T]$:

$$Y_s^{v,t,x}=H(X_T^{v,t,x})+\int_s^T g(r,X_r^{v,t,x},Y_r^{v,t,x},Z_r^{v,t,x})\,\mathrm{d}r-\int_s^T Z_r^{v,t,x}\,\mathrm{d}B_r.$$

Therefore, for every $(t,x)\in[0,T]\times\mathbf{R}^P$, the triple $(X^{v,t,x},Y^{v,t,x},Z^{v,t,x})$ satisfies the FBSDE (E) associated to the initial condition (t,x). Hence, from Theorem 1.1, for every $(t,x)\in[t_{N-1},T]\times\mathbf{R}^P$, $u(t,x)=v(t,x)$. Using an induction as described in Section 2, we complete the proof. □

Note that we have not been able to propose a purely probabilistic approach of this problem, in the sense that we do not know how to prove by means of stochastic tools the estimate of the gradient given by Ladyzenskaja et al.

Nevertheless, to close this paper, we want to show, by means of the arguments developed in this appendix, an existence and unique result in a degenerate case, under the condition $P=Q=1$. Actually, the point concerning existence was already mentioned in Hu and Yong (2000a, b). Hu (2000a, b).

Appendix C. A degenerate case, $P = Q = 1$

Assumption (C.A3). We say that the functions g, h and σ satisfy Assumption (C.A3), if there exists three constants k, K and Λ such that they satisfy the following properties:

(C.A3.0): $g:[0,T]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

$\sigma:[0,T]\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$,

$h:\mathbf{R}\to\mathbf{R}$.

(C.A3.1): $\forall t\in[0,T]$, $\forall(x,x',y,y')\in\mathbf{R}^4$,

$$|g(t,x',y')-g(t,x,y)|\leqslant k(|x-x'|+|y-y'|);$$
$$|h(x')-h(x)|\leqslant k|x-x'|.$$

(C.A3.2): g, h and σ satisfy Assumption (A1) with respect to K and Λ.

Theorem C.1. *Under Assumption* (C.A3), *and for every* $T>0$, *the problem* (E) *associated to the coefficients* (g,h,σ) *admits a unique solution. Moreover, there* *exists a constant* $\Gamma^{(C)}$, *only depending on* k *and* T (*and not on* K), *such that*

$$\forall (t,x)\in[0,T]\times\mathbf{R}^P,\quad |\partial_x Y_t^{t,x}|\leqslant \Gamma^{(C)}. \tag{C.1.1}$$

Proof. Let us firstly assume that (g,h,σ) satisfy Assumption (B.A2). From the first part, we know that there exists a constant $\delta>0$, only depending on K, such that, for every $t\in[T-\delta,T]$, $\forall x\in\mathbf{R}^P$, the problem:

$$\begin{cases} \forall s\in[t,T],\\ X_s^{t,x}=x+\displaystyle\int_t^s \sigma(u,X_u^{t,x},Y_u^{t,x})\,\mathrm{d}B_u,\\ Y_s^{t,x}=h(X_T^{t,x})+\displaystyle\int_s^T g(u,X_u^{t,x},Y_u^{t,x})\,\mathrm{d}u-\int_s^T Z_u^{t,x}\,\mathrm{d}B_u,\\ \mathbf{E}\displaystyle\int_t^T (|X_s^{t,x}|^2+|Y_s^{t,x}|^2+|Z_s^{t,x}|^2)\,\mathrm{d}s<\infty \end{cases}$$

admits a unique solution, denoted $(X_s^{t,x},Y_s^{t,x},Z_s^{t,x})_{t\leqslant s\leqslant T}$. Hence, we can define

$$\theta:[T-\delta,T]\times\mathbf{R}\to\mathbf{R},\ (t,x)\mapsto Y_t^{t,x}.$$

Modifying δ if necessary, we know from Section 1 and from Appendix B that the map θ belongs to $\mathbf{C}^{1,2}([T-\delta,T]\times\mathbf{R},\mathbf{R})$ and that its partial derivative of order one with respect to x is bounded. Moreover, the processes $(X_s^{t,x})_{t\leqslant s\leqslant T}$, $(Y_s^{t,x})_{t\leqslant s\leqslant T}$ and $(Z_s^{t,x})_{t\leqslant s\leqslant T}$ are a.s. differentiable with respect to x and the process $(\partial_x X_s^{t,x},\partial_x Y_s^{t,x},\partial_x Z_s^{t,x})_{t\leqslant s\leqslant T}$ satisfies the following system:

$$\begin{cases} \forall s\in[t,T],\\ \partial_x X_s^{t,x}=1+\displaystyle\int_t^s\left((\sigma'_x(u,X_u^{t,x},Y_u^{t,x})+\sigma'_y(u,X_u^{t,x},Y_u^{t,x})\frac{\partial\theta}{\partial x}(u,X_u^{t,x}))\partial_x X_u^{t,x}\right)\mathrm{d}B_u,\\ \partial_x Y_s^{t,x}=h'(X_T^{t,x})\partial_x X_T^{t,x}+\displaystyle\int_t^T (g'_x(u,X_u^{t,x},Y_u^{t,x})\partial_x X_u^{t,x}\\ \qquad +g'_y(u,X_u^{t,x},Y_u^{t,x})\partial_x Y_u^{t,x})\,\mathrm{d}u-\displaystyle\int_s^T \partial_x Z_u^{t,x}\,\mathrm{d}B_u. \end{cases}$$

Noting that the map $\partial\theta/\partial x$ is bounded, we see that $(\partial_x X_s^{t,x})_{t\leqslant s\leqslant T}$ is an exponential martingale, and therefore, we have a.s.:

$$\forall s\in[t,T],\ \partial_x X_s^{t,x}>0$$

and,

$$\mathbf{E}\,|\partial_x X_s^{t,x}|=\mathbf{E}\partial_x X_s^{t,x}=1.$$

Therefore, using Itô's calculus, $\forall s \in [T-\delta, T]$

$$\mathbf{E}\sqrt{1+|\partial_x Y_s^{t,x}|^2} \leqslant \mathbf{E}\sqrt{1+|\partial_x Y_T^{t,x}|^2} + \mathbf{E}\int_s^T (|g_x'(u, X_u^{t,x}, Y_u^{t,x})|\partial_x X_u^{t,x} + |g_y'(u, X_u^{t,x}, Y_u^{t,x})|\sqrt{1+|\partial_x Y_u^{t,x}|^2})\,\mathrm{d}u.$$

From Assumption (C.A3), and applying Gronwall's lemma, we prove that there exists a constant $\Gamma^{(6)}$, only depending on k and T, such that $\forall s \in [T-\delta, T]$,

$$\mathbf{E}\sqrt{1+|\partial_x Y_s^{t,x}|^2} \leqslant \Gamma^{(6)}.$$

In particular,

$$|\partial_x Y_t^{t,x}| \leqslant \Gamma^{(6)}.$$

Therefore, the map θ is $\Gamma^{(6)}$-lipschitzian. So, doing the same kind of iteration as done in the second part, we prove that the problem $(\mathbf{E})$ admits a unique solution. This one satisfies the property (C.1.1).

Using Corollary (1.7), we prove that Theorem A.1 is still true if (g, h, σ) satisfy Assumption (C.A3). $\square$

References

Antonelli, F., 1993. Backward–forward stochastic differential equations. Ann. Appl. Probab. 3, 777–793.

Hu, Y., 2000a. On the existence of solutions to one-dimensional forward–backward SDEs. Stochastic Anal. Appl. 18, 101–111.

Hu, Y., 2000b. On the solution of forward–backward SDEs with monotone and continuous coefficients. Non-linear Anal. 42, 1–12.

Hu, Y., Peng, S., 1995. Solution of forward–backward stochastic differential equations. Probab. Theory Related Fields 103, 273–283.

Hu, Y., Yong, J., 2000. Forward–backward stochastic differential equations with nonsmooth coefficients. Stochastic Process. Appl. 87, 93–106.

Jacod, J., Shiryaev, A.N., 1987. Limit theorems for stochastic processes. Grundlehren der mathematischen Wissenschaften. Springer, Berlin, Heidelberg.

Karatzas, I., Shreve, S.E., 1988. Brownian motion and stochastic calculus. Graduate Texts in Mathematics. Springer, New York.

Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N., 1968. Linear and quasi-linear equations of parabolic type. Translation of Mathematical Monographs. American Mathematical Society, Providence, RI.

Ma, J., Yong, J., 1995. Solvability of forward–backward SDEs and the nodal set of Hamilton–Jacobi–Bellman equations. Chinese Ann. Math. Ser. B 16, 279–298.

Ma, J., Yong, J., 1999. Forward–Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Mathematics, Vol. 1702. Springer, Berlin.

Ma, J., Protter, P., Yong, J., 1994. Solving forward–backward stochastic differential equations explicitly—a four step scheme. Probab. Theory Related Fields 98, 339–359.

Nualart, D., 1995. The Malliavin Calculus and Related Topics. Springer, New York.

Pardoux, É., 1999. BSDE's, weak convergence and homogenization of semilinear PDE's. In: Clarke, F.H., Stern, R.J. (Eds.). Nonlinear Analysis, Differential Equations and Control. Kluwer Academic Publishers, pp. 503–549.

Pardoux, É., Peng, S., 1992. Backward stochastic differential equations and quasi-linear parabolic partial differential equations. In: Rzosvskii, B.L., Sowers, R.S. (Eds.), Stochastic Partial Differential Equations and their Applications, Lecture Notes in Control & Information Sciences, Vol. 176. Springer, Berlin, Heidelberg, New York, pp. 200–217.

Pardoux, É., Tang, S., 1999. Forward–backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Related Fields 114, 123–150.

Peng, S., Wu, Z., 1999. Fully coupled forward–backward stochastic differential equations and applications to optimal control. SIAM J. Control Optim. 37 (3), 825–843.

Rogers, L.C.G., Williams, D., 1987. Diffusions, Markov Processes and Martingales: Itô Calculus. Wiley, Chichester, UK.

Yong, J., 1997. Finding adapted solutions of forward–backward stochastic differential equations—method of continuation. Probab. Theory Related Fields 107, 537–572.

Yong, J., 1999. Linear forward–backward stochastic differential equations. Appl. Math. Optim. 39, 93–119.